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A SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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We define a linear connection on a Riemannian manifold which is semi-symmetric but non-metric and study some properties of the curvature tensor and Weyl projective curvature tensor with respect to semi-symmetric non-metric connection.

1. INTRODUCTION

Friedmann and Schouten[1,2] introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden[3] introduced semi-symmetric metric connection on a Riemannian manifold and this was further developed by Yano[4], Imai[5,6] Nakao[7] and Amur and Pujar[8].

In this paper, we define a semi-symmetric non-metric connection $\overset{*}{\nabla}$ on a Riemannian manifold M and define the curvature tensor of M with respect to semi-symmetric non-metric connection. We obtain a relation connecting the curvature tensors of M with respect to semi-symmetric non-metric connection and the Riemannian connection. Further, we obtain the first and second Bianchi identities associated with the semi-symmetric non-metric connection.

We also define Weyl projective curvature tensor with respect to semi-symmetric non-metric connection and show that it is equal to Weyl projective curvature tensor with respect to the Riemannian connection.

2. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let M be an m-dimensional Riemannian manifold with Riemannian metric g.

We define a linear connection $\overset{*}{\nabla}$ on a Riemannian manifold M by

$$\overset{*}{\nabla}_X Y = \nabla_X Y + u(Y)X \qquad \text{...(2.1)}$$

where ∇ is the Riemannian connection on M and u is a 1-form associated with the vector field U on M by

$$u(X) = g(U,X). \qquad \text{...(2.2)}$$

Using (2.1), the torsion tensor $\overset{*}{T}$ of M with respect to the connection $\overset{*}{\nabla}$ is given by

$$\begin{aligned}\overset{*}{T}(X,Y) &= \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X,Y]\\ &= u(Y)X - u(X)Y. \qquad \text{...(2.3)}\end{aligned}$$

A linear connection satisfying (2.3) is called a semi-symmetric connection[1,2,9]

Further, using (2.1), we have

$$\overset{*}{\nabla}_X(g(Y,Z)) = (\overset{*}{\nabla}_X g)(Y,Z) + g(\overset{*}{\nabla}_X Y,Z) + g(Y, \overset{*}{\nabla}_X Z)$$

$$= (\overset{*}{\nabla}_X g)(Y,Z) + \nabla_X (g(Y,Z))$$

$$+ u(Y)g(X,Z) + u(Z)g(X,Y)$$

which implies

$$(\overset{*}{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y) \qquad \text{...(2.4)}$$

for vector fields X, Y, Z on M.

A linear connection $\overset{*}{\nabla}$ defined by (2.1) satisfies (2.3) and (2.4) and hence we call $\overset{*}{\nabla}$, a semi-symmetric non-metric connection.

Conversely, we will show that a linear connection satisfying (2.3) and (2.4) is defined by (2.1).

Let $\overset{*}{\nabla}$ be a linear connection defined on M by

$$\overset{*}{\nabla}_X Y = \nabla_X Y + H(X,Y). \qquad \text{...(2.5)}$$

where ∇ is the Riemannian connection and H is a tensor of type (1, 2) defined on M and $\overset{*}{\nabla}$ satisfies (2.3) and (2.4).

From (2.4) and (2.5), we have

$$\overset{*}{\nabla}_X(g(Y,Z)) = g(\overset{*}{\nabla}_X Y,Z) + g(Y, \overset{*}{\nabla}_X Z) - u(Y)g(X,Z) - u(Z)g(X,Y)$$

$$= \nabla_X(g(Y,Z)) + g(H(X,Y),Z) + g(H(X,Z),Y)$$

$$- u(Y)g(X,Z) - u(Z)g(X,Y).$$

which implies

$$g(H(X,Y),Z) + g(H(X,Z),Y) = u(Y)g(X,Z) + u(Z)g(X,Y). \qquad \text{...(2.6)}$$

On the other hand, from (2.5), we have

$$\overset{*}{T}(X,Y) = H(X,Y) - H(Y,X). \qquad \text{...(2.7)}$$

From (2.2), (2.6) and (2.7), we have

$$g(\overset{*}{T}(X,Y),Z) + g(\overset{*}{T}(Z,X),Y) + g(\overset{*}{T}(Z,Y),X)$$

$$= g(H(X,Y),Z) - g(H(Y,X),Z) + g(H(Z,X),Y)$$

$$- g(H(X,Z),Y) + g(H(Z,Y),X) - g(H(Y,Z),X)$$

$$= 2\{g(H(X,Y),Z) - u(Z)g(X,Y)\}$$

$$= 2\{g(H(X,Y),Z) - g(Z,U)g(X,Y)\}. \qquad \text{...(2.8)}$$

Hence we obtain

$$H(X,Y) = \frac{1}{2}\{\overset{*}{T}(X,Y) + \overset{*}{T}'(X,Y) + \overset{*}{T}'(Y,X)\} + g(X,Y)U. \quad \text{...(2.9)}$$

where the tensor $\overset{*}{T}'$ of type (1, 2) is defined on M by

$$g(\overset{*}{T}(Z,X),Y) = g(\overset{*}{T}'(X,Y), Z). \quad \text{...(2.10)}$$

From (2.3) and (2.10), we have

$$\begin{aligned} g(\overset{*}{T}'(X,Y),Z) &= g(u(X)Z - u(Z)X, Y) \\ &= u(X)g(Z,Y) - u(Z)g(X,Y) \\ &= u(X)g(Z,Y) - g(Z,U)g(X,Y). \end{aligned} \quad \text{...(2.11)}$$

which implies

$$\overset{*}{T}'(X,Y) = u(X)Y - g(X,Y)U. \quad \text{...(2.12)}$$

Hence from (2.3), (2.9) and (2.12), we have

$$\begin{aligned} H(X,Y) &= \frac{1}{2}\{u(Y)X - u(X)Y + u(X)Y - g(X,Y)U \\ &\qquad + u(Y)X - g(Y,X)U\} + g(X,Y)U \\ &= u(Y)X \end{aligned} \quad \text{...(2.13)}$$

on M.

Hence from (2.5) and (2.13), we obtain

$$\overset{*}{\nabla}_X Y = \nabla_X Y + u(Y)X.$$

Further, for a 1-form π on M, we have

$$\begin{aligned} \overset{*}{\nabla}_X(\pi(Y)) &= (\overset{*}{\nabla}_X\pi)Y + \pi(\overset{*}{\nabla}_X Y) \\ &= (\overset{*}{\nabla}_X\pi)Y + \pi(\nabla_X Y) + u(Y)\pi(X) \\ &= (\overset{*}{\nabla}_X\pi)Y + \nabla_X(\pi(Y) - (\nabla_X\pi)Y + u(Y)\pi(X) \end{aligned}$$

which implies

$$(\overset{*}{\nabla}_X\pi)Y = (\nabla_X\pi)Y - u(Y)\pi(X) \quad \text{...(2.14)}$$

for vector fields X, Y on M.

Convariant differentiation of the torsion tensor $\overset{*}{T}$is given by

$$\begin{aligned} (\overset{*}{\nabla}_X \overset{*}{T})(Y,Z) &= \overset{*}{\nabla}_X(\overset{*}{T}(Y,Z)) - \overset{*}{T}(\overset{*}{\nabla}_X Y, Z) - \overset{*}{T}(Y, \overset{*}{\nabla}_X Z) \\ &= ((\overset{*}{\nabla}_X u)Z)Y - ((\overset{*}{\nabla}_X u)Y)Z. \end{aligned} \quad \text{...(2.15)}$$

Further we define

$$\overset{*}{T}(X, Y, Z) = g(\overset{*}{T}(X, Y), Z). \qquad \ldots(2.16)$$

From (2.3) and (2.16), we have

$$\overset{*}{T}(X, Y, Z) + \overset{*}{T}(Y, Z, X) + \overset{*}{T}(Z, X, Y) = 0. \qquad \ldots(2.17)$$

This identity is true for any semi-symmetric connection on M.

3. CURVATURE TENSOR OF M WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

Analogous to the definition of curvature tensor of a Riemannian manifold M with respect to the Riemannian connection ∇, we define the curvature tensor of M with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ by

$$\overset{*}{R}(X, Y) Z = \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z - \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z - \overset{*}{\nabla}_{[X, Y]} Z. \qquad \ldots(3.1)$$

From (2.5) and (3.1), we have

$$\begin{aligned}\overset{*}{R}(X, Y) Z &= \overset{*}{\nabla}_X (\nabla_Y Z + u(Z) Y) - \overset{*}{\nabla}_Y (\nabla_X Z + u(Z) X) \\ &\quad - \nabla_{[X, Y]} Z - u(Z) [X, Y] \\ &= \nabla_X (\nabla_Y Z + u(Z) Y) + u(\nabla_Y Z + u(Z) Y) X \\ &\quad - \nabla_Y (\nabla_X Z + u(Z) X) - u(\nabla_X Z + u(Z) X) Y \\ &\quad - \nabla_{[X, Y]} Z - u(Z) [X, Y] \\ &= R(X, Y) Z + s(X, Z) Y - s(Y, Z) X \qquad \ldots(3.2)\end{aligned}$$

where

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

is the curvature tensor of a Riemannian manifold M with respect to the Riemannian connection ∇ and s is a tensor of type (0, 2) defined on M by

$$\begin{aligned} s(X, Y) &= (\nabla_X u) Y - u(X) u(Y) \\ &= (\overset{*}{\nabla}_X u) Y. \qquad \ldots(3.3)\end{aligned}$$

A relation between the curvature tensors of M with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the Riemannian connection ∇ is given by (3.2).

Using (3.3), we have

$$\begin{aligned} s(X, Y) - s(Y, X) &= (\overset{*}{\nabla}_X u) Y - (\overset{*}{\nabla}_Y u) X \\ &= Xu(Y) - Yu(X) - u[X, Y] \\ &= du(X, Y). \qquad \ldots(3.4)\end{aligned}$$

Thus a tensor s is symmetric iff the 1-form u is closed.

Let

$$R(X, Y; Z, W) = g(R(X, Y)Z, W)$$

and

$$\overset{*}{R}(X, Y; Z, W) = g(\overset{*}{R}(X, Y)Z, W) \qquad \ldots(3.5)$$

for vector fields X, Y, Z, W on M.

From (3.2) and (3.5), we have

$$\overset{*}{R}(X, Y; Z, W) = R(X, Y; Z, W) + s(X, Z)\, g(Y, W) - s(Y, Z)\, g(X, W). \qquad \ldots(3.6)$$

From (3.1) and (3.5), we have

$$\overset{*}{R}(X, Y; Z, W) + \overset{*}{R}(Y, X; Z, W) = 0. \qquad \ldots(3.7)$$

Using (3.2), (3.6) and the first Bianchi identity with respect to the Riemannian connection, we have

$$\begin{aligned}\overset{*}{R}(X, Y)Z + \overset{*}{R}(Y, Z)X + \overset{*}{R}(Z, X)Y = {}& \{s(Z, Y) - s(Y, Z)\}X \\ &+ \{s(X, Z) - s(Z, X)\}Y \\ &+ \{s(Y, X) - s(X, Y)\}Z\end{aligned} \qquad \ldots(3.8)$$

and hence

$$\begin{aligned}\overset{*}{R}(X,Y;Z,W) + \overset{*}{R}(Y,Z;X,W) + \overset{*}{R}(Z,X;Y,W) = {}& \{s(Z,Y) - s(Y,Z)\}\, g(X,W) \\ &+ \{s(X,Z) - s(Z,X)\}\, g(Y,W) \\ &+ \{s(Y,X) - s(X,Y)\}\, g(Z,W).\end{aligned}$$

We call (3.8) as the first Bianchi identity with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$.

In particular, if the 1-form u is closed, then (3.8) reduces to

$$\overset{*}{R}(X, Y)Z + \overset{*}{R}(Y, Z)X + \overset{*}{R}(Z, X)Y = 0. \qquad \ldots(3.9)$$

Using (3.6), we have

$$\begin{aligned}\overset{*}{R}(X, Y; Z, W) + \overset{*}{R}(X, Y; W, Z) = {}& s(X, Z)\, g(Y, W) + s(X, W)\, g(Y, Z) \\ &- s(Y, Z)\, g(X, W) - s(Y, W)\, g(X, Z)\end{aligned} \qquad (3.10)$$

and

$$\begin{aligned}\overset{*}{R}(X, Y; Z, W) - \overset{*}{R}(Z, W; X, Y) = {}& \{s(X, Z) - s(Z, X)\}\, g(Y, W) \\ &+ s(W,X)\, g(Y,Z) - s(Y,Z)\, g(X,W).\end{aligned} \qquad (3.11)$$

Using (2.1), (3.7) and the second Bianchi Identity for the Riemannian connection, we obtain the second Bianchi identity associated with semi-symmetric non-metric connection which is given by

$$\begin{aligned}(\overset{*}{\nabla}_X \overset{*}{R})\,(Y, Z) + (\overset{*}{\nabla}_Y \overset{*}{R})\,(Z, X) + (\overset{*}{\nabla}_Z \overset{*}{R})\,(X, Y) \\ = -\overset{*}{R}\,(\overset{*}{T}\,(X, Y), Z) - \overset{*}{R}\,(\overset{*}{T}\,(Y, Z), X) - \overset{*}{R}\,(\overset{*}{T}\,(Z, X), Y) \\ = 2\,\{u\,(X)\,\overset{*}{R}\,(Y, Z) + u(Y)\,\overset{*}{R}\,(Z, X) + u(Z)\,\overset{*}{R}\,(X, Y)\}.\end{aligned} \qquad \ldots(3.12)$$

Analogous to the definition of Ricci tensor of a Riemannian manifold M with respect to the Riemannian connection ∇, we define Ricci tensor of M with respect to semi-symmetiric non-metric connection $\overset{*}{\nabla}$ by

$$\overset{*}{Ric}\,(Y, Z) = \sum_{i=1}^{m} \overset{*}{R}(E_i, Y; Z, E_i) \qquad \ldots(3.13)$$

where E_i's, $1 \leq i \leq m$, are orthonormal vector fields on M. From (3.5) and (3.13), we have

$$\overset{*}{Ric}\,(Y, Z) = Ric\,(Y, Z) - (m - 1)\, s\,(Y, Z) \qquad \ldots(3.14)$$

where

$$Ric\,(Y, Z) = \sum_{i=1}^{m} R(E_i, Y; Z, E_i)$$

is Ricci tensor of M with respect to the Riemannian connection.

A relation between Ricci tensors with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the Riemannian connection ∇ is given by (3.14).

Further, if $\overset{*}{Ric}\,(X, Y) = 0$ on M, then (3.14) implies s is symmetric.

From (3.4) and (3.14), we have

$$\overset{*}{Ric}\,(X, Y) - \overset{*}{Ric}\,(Y, X) = -\,(m - 1)\, du\,(X, Y).$$

Hence, Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ is symmetric iff the 1-form u is closed and hence iff s is symmetric.

Using (2.15) and (3.3), we have

$$(\overset{*}{\nabla}_X \overset{*}{T})\,(Y, Z) = s(X, Z)\, Y - s(X, Y)\, Z. \qquad \ldots(3.15)$$

In particular, if either the 1-form u is closed or Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ vanishes, then from (3.15), we have

$$(\overset{*}{\nabla}_X \overset{*}{T})\,(Y, Z) + (\overset{*}{\nabla}_Y \overset{*}{T})\,(Z, X) + (\overset{*}{\nabla}_Z \overset{*}{T})\,(X, Y) = 0. \qquad \ldots(3.16)$$

Analogous to the definition of the scalar curvature of a Riemannian manifold M with respect to the Riemannian connection, we define the scalar curvature of M with respect to semi-symmetric non-metric connection by

$$\overset{*}{r} = \sum_{i=1}^{m} \overset{*}{Ric}\,(E_i, E_i). \qquad \ldots(3.17)$$

From (3.14) and (3.17), we obtain a relation between the scalar curvature of M with respect to the Riemannian connection and the semi-symmetric non-metric connection which is given by

$$\overset{*}{r} = r - (m - 1) \text{ trace } S \qquad \ldots(3.18)$$

where

$$r = \sum_{i=1}^{m} Ric\,(E_i, E_i)$$

is the scalar curvature of M with respect to the Riemannian connection and S is a tensor of type (1, 1) defined on M by

$$s\,(X, Y) = g(SX, Y).$$

4. PROJECTIVE CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

Weyl projective curvature tensor of a Riemannian manifold M with respect to the Riemannian connection is given by

$$P(X, Y)\,Z = R(X, Y)\,Z - \frac{1}{m-1}\{(\overset{*}{Ric}\,(Y,Z)\,X - \overset{*}{Ric}\,(X,Z)Y)\}. \qquad \ldots(4.1)$$

Analogous to this definition, we define projective curvature tensor of M with respect to semi-symmetric non-metric connection by

$$\overset{*}{P}\,(X,Y)Z=\overset{*}{R}\,(X,Y)Z - \frac{1}{m-1}\{(\overset{*}{Ric}\,(Y,Z)\,X - \overset{*}{Ric}\,(X,Z)Y)\}. \qquad \ldots(4.2)$$

From (3.2), (3.14), (4.1) and (4.2), we have

$$\overset{*}{P}(X, Y)\,Z = P(X, Y)\,Z \text{ on } M. \qquad \ldots(4.3)$$

Hence we have

Theorem 4.1 — If M is a Riemannian manifold admitting semi-symmetric non-metric connection, then the Weyl projective curvature tensor with respect to semi-symmetric non-metric connection is equal to the Weyl projective curvature tensor with respect to the Riemannian connection.

From (4.3), we have, the projective curvature tensor with respect to semi-symmetric non-metric connection satisfies the following algebraic properties

$$\overset{*}{P}\,(X, Y)\,Z + \overset{*}{P}\,(Y, X)\,Z = 0$$

and

$$\overset{*}{P}\,(X, Y)\,Z + \overset{*}{P}\,(Y, Z)\,X + \overset{*}{P}\,(Z, X)\,Y = 0 \qquad \ldots(4.4)$$

for vector fields X, Y, Z on M.

In particular, let M be a Riemannian manifold satisfying

$$\overset{*}{R}\,(X, Y)\,Z = 0. \qquad \ldots(4.5)$$

which implies

$$\overset{*}{Ric}\ (Y, Z) = 0 \text{ on } M. \qquad \ldots(4.6)$$

From (4.2), (4.3), (4.5) and (4.6), we have

$$P\ (X, Y)\ Z = 0 \text{ on } M.$$

Necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold[10-13].

From (3.3), (3.14) and (4.6), we have

$$(\nabla_X u)\ Y = \frac{1}{m-1}\ Ric\ (X, Y) + u(X)\ u(Y). \qquad \ldots(4.7)$$

Using (4.7), we have

$$\begin{aligned} -u\ (R(X, Y)Z) &= (\nabla_X \nabla_Y u - \nabla_Y \nabla_X u - \nabla_{[X,Y]} u)\ Z \\ &= \frac{1}{m-1}\ ((\nabla_X Ric)\ (Y, Z) - (\nabla_Y Ric)\ (X, Z) \\ &\quad + u(Y)\ Ric\ (X, Z) - u(X)\ Ric\ (Y, Z)). \qquad \ldots(4.8) \end{aligned}$$

Further, from (4.1), we have

$$u(R(X, Y)Z) = \frac{1}{m-1}\ (Ric\ (X, Z)u(X) - Ric(X, Z)u(Y)). \ldots(4.9)$$

From (4.8) and (4.9), we have

$$(\nabla_X Ric)\ (Y, Z) - (\nabla_Y Ric)\ (X, Z) = 0.$$

Hence we have

Theorem 4.2 — If M is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is projectively flat and

$$(\nabla_X Ric)\ (Y, Z) = (\nabla_Y Ric)\ (X, Z) \text{ on } M.$$

It is well known that a Riemannian manifold is of constant curvature iff it is projectively flat and a Riemannian manifold of constant curvature is conformally flat[13].

Hence from theorem 4.2, we have

Theorem 4.3 — If M is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is a space of constant curvature and hence is conformally flat.

From (3.2), (3.15), (4.5) and (4.6), we have

$$\begin{aligned} R\ (X, Y)\ Z &= s\ (Z, Y)\ X - s\ (Z, X)\ Y \\ &= (\overset{*}{\nabla}_Z \overset{*}{T})\ (X, Y). \qquad \ldots(4.10) \end{aligned}$$

Hence we have

Theorem 4.4 — If M is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is flat iff

$$(\overset{*}{\nabla}_X T)\ (Y, Z) = 0 \text{ on } M.$$

A Riemannian manifold M is a group manifold[14] with respect to semi-symmetric non-metric connection if

$$\overset{*}{R}\ (X, Y)\ Z = 0$$

and

$$(\overset{*}{\nabla}_X \overset{*}{T})\ (Y, Z) = 0 \quad \text{on } M. \qquad \text{...(4.11)}$$

Hence from (4.11) and Theorem 4.4, we have

Theorem 4.5 – If a Riemannian manifold M is a group manifold with respect to semi-symmetric non-metric connection, then M is flat and consequently M is projectively flat and conformally flat.

In particular, if either the 1-form u is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then from (3.2) and (3.15), we have

$$\overset{*}{R}\,(X, Y)\ Z = R\ (X, Y)\ Z + (\overset{*}{\nabla}_Z \overset{*}{T})\ (Y, X). \qquad \text{...(4.12)}$$

Hence we have

Theorem 4.6 – Let M be a Riemannian manifold with semi-symmetric non-metric connection. If either the 1-from u is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then

$$R\ (X, Y)\ Z = \overset{*}{R}\ (X, Y)\ Z + (\overset{*}{\nabla}_Z \overset{*}{T})\ (X, Y) \text{ on } M.$$

In particular, if M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then from (4.2), (4.3) and (4.12), we have

$$\begin{aligned} P\ (X, Y)\ Z &= \overset{*}{R}\,(X, Y)\ Z \\ &= R\ (X, Y)\ Z + (\overset{*}{\nabla}_Z \overset{*}{T})\ (Y, X) \text{ on } M. \end{aligned} \qquad \text{...(4.13)}$$

Hence we have :

Theorem 4.7 – If a Riemannian manifold M with vanishing Ricci tensor with respect to semi-symmetric non-metric connection is projectively flat, then the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

From Theorem 4.2 and 4.7, we have :

Theorem 4.8 – If M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then M is projectively flat iff the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

Since a flat manifold is projectively flat, from (4.13) we have

$$\overset{*}{R}\,(X, Y)\ Z = 0 \text{ and } (\overset{*}{\nabla}_Z \overset{*}{T})\ (X, Y) = 0 \text{ on } M.$$

Hence from (4.11), we have

Theorem 4.9 – If a Riemannian manifold M with vanishing Ricci tensor with respect to semi-symmetric non-metric connection is flat, then M is a group manifold with respect to semi-symmetric non-metric connection.

From Theorem 4.5 and 4.9, we have

Theorem 4.10 – If M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then M is a group manifold with respect to semi-symmetric non-metric connection iff M is flat.

Let ∇^p be a symmetric linear connection which is projectively related to the Riemannian connection ∇ defined on M by[12,13]

$$\overset{p}{\nabla}_X Y = \nabla_X Y + u(Y)\, X + u(X)\, Y. \qquad \text{...(4.14)}$$

From (2.1) and (4.14), we have

$$\overset{p}{\nabla}_X Y = \overset{*}{\nabla}_X Y + u(X)\, Y \text{ on } M. \qquad \text{...(4.15)}$$

A relation between the curvature tensors of M with respect to ∇^p and the Riemannian connection ∇ is given by[12]

$$R^p\,(X, Y)\, Z = R\,(X, Y)\, Z + s(X, Z)\, Y - s(Y, Z)\, X$$
$$+ \{s(X, Y) - s(Y,X)\}\, Z. \qquad \text{...(4.16)}$$

Hence from (3.2) and (4.16), we obtain

$$R^p\,(X, Y)\, Z = \overset{*}{R}\,(X, Y)\, Z + \{s(X, Y) - s\,(Y, X)\}\, Z. \qquad \text{...(4.17)}$$

which is a relation between the curvature tensors of M with respect to ∇^p and semi-symmetric non-metric connection $\overset{*}{\nabla}$.

In particular, let M be a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection. Then s is symmetric.

From (4.13) and (4.17), we have

$$R^p\,(X, Y)\, Z = \overset{*}{R}\,(X, Y)\, Z = P\,(X, Y)\, Z \text{ on } M. \qquad \text{...(4.18)}$$

Hence we have :

Theorem 4.11 – If M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then there exists a symmetric linear connection which is projectively related to the Riemannian connection and its curvature tensor is equal to the Weyl projective curvature tensor with respect to the Riemannian connection.

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