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H^1 BESSEL EXPANSIONS

ALLAN M. KRALL

Department of Mathematics, The Pennsylvania State University, McAllister Building, University Park, PA 16802, *USA*

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Traditionally Bessel functions are considered in the spaces $L^2(0, 1; x)$ and $L^2(0, \infty; x)$, where the weight function x is the coefficient of λ in the formally self-adjoint differential equation

$$-(xy')' + (n^2/x)y = \lambda xy,$$

satisfied by the Bessel functions $J_n(\sqrt{\lambda}\, x)$, $J_{-n}(\sqrt{\lambda}\, x)$ and/or $Y_n(\sqrt{\lambda}\, x)$. We examine instead the same equation and its solutions in a Sobolev spaces H^1, generated in part by the left side of the differential equation. The Bessel operators remain self-adjoint and their spectral resolutions remain the same in the new settings.

1. INTRODUCTION

Bessel functions are solutions to the differential equation $ly = \lambda y$, where the expression l is given by

$$ly = \{(-(xy')' + [(n^2/x) + 1x]\, y\}/x.$$

(We have added 1 to the left side of $ly = \lambda y$ so that ultimately 0 will be in the resolvent sets of formal operators to be defined shortly.) Traditionally set in $L^2(0, 1; x)$, generated by the inner product

$$\langle f, g \rangle_{L^2} = \int_0^1 f\bar{g}\, x\, dx$$

or in $L^2(0, \infty; x)$, generated by the inner product

$$\langle f, g \rangle_{L^2} = \int_0^\infty f\bar{g}\, x\, dx$$

the expression l, together with appropriate boundary conditions, is used to define a self-adjoint operator in each setting. On the finite interval the operator has a spectral resolution which is the traditional Bessel eigenfunction expansion, while on the infinite interval the spectral resolution is the inverse of the Hankel transform.

We shall show that these results carry over to new spaces generated by the expression l itself. Since such spaces not only involve f and g in the inner product, but also include f' and g', they are Sobolev spaces. The key to the extension involves a singular Dirichlet integral.

We examine the intervals $(0, 1]$ and $(0, \infty)$ each in turn. For technical reasons we shall address later, we restrict our attention to the cases $n \geq 0$.

2. The Interval (0, 1]

We shall first review briefly the classical facts concerning the Bessel operator on $(0, 1]$, giving its domain, form, eigenfunction-eigenvalues and its spectral resolution. We then establish the Dirichlet integral connecting $L^2(0, 1; x)$ and the accompanying Sobolev space. We finally prove that the Bessel operator, suitably restricted, remains self-adjoint with the same Green's function, eigenfunction-eigenvalues and spectral resolution.

Definition 2.1 — We denote by $D_{(0,1]}$ those elements y in $L^2(0, 1; x)$ which satisfy

(1) y is absolutely continuous on all closed subsets of $(0, 1]$.

(2) (xy') is absolutely continuous on all closed subsets of $(0, 1]$..

(3) ly exists a.e. and is in $L^2(0, 1; x)$.

(4) If $n = 0$, y satisfies

$$\lim_{x \to 0} -xy'(x) = 0.$$

If $n > 0$, y satisfies

$$\lim_{x \to 0} x(nx^{n-1} y(x) - x^n y'(x)) = 0$$

(5) for $0 \leq \delta \leq \pi/2$,

$$\cos \delta\, y(1) + \sin \delta\, y'(1) = 0.$$

We define the Bessel operator L by setting $Ly = ly$ for all y in $D_{(0,1]}$.

The theory of boundary values for singular operators[2,3,5] shows that for $0 \leq n < 1$, the boundary condition at 0 is needed (limit circle case), but for $1 \leq n < \infty$, it is automatic (limit point case).

The solution of $ly = \lambda y$ satisfying the boundary condition at $x = 0$ is $J_n(\sqrt{\lambda - 1}\, x)$. Eigenvalues for L are therefore given by the zeros $\{\lambda_k\}_{k=1}^{\infty}$ of

$$\left[\cos \delta\, J_n(\sqrt{\lambda - 1}\, x) + \sin \delta \frac{d}{dk} J_n(\sqrt{\lambda - 1}\, x)\right]_{x=1} = 0.$$

The spectrum of L consists only of these real eigenvalues.

Further, if the differential equation

$$-(xy')' + (n^2/x)y = (\lambda - 1)\,xy$$

is multiplied by y, and integration is carried out over $(0, 1]$, the result

$$\frac{\cot\delta\, y(1)^2 + \int_0^1 (y')^2\, x\, dx + \int_0^1 (n^2 y^2/x)\, dx}{\int_0^1 y^2\, x\, dx} = \lambda - 1$$

shows that $\lambda \geq 1$. $(y = J_n(\sqrt{\lambda - 1}\, x))$.

What is more, if the differential equation is multiplied by xy' and integration over $(0, 1]$ is performed, the result when $\delta \neq 0$,

$$\int_0^1 y^2\, x\, dx = y^2(1)\,[\lambda - 1 + \cot^2\delta]/2\,(\lambda - 1),$$

gives a formula for the norm square of the Bessel function eigenfunctions,

$$\| J_n(\sqrt{\lambda_k - 1}\, x) \|^2 = J_n(\sqrt{\lambda_k - 1}\, x)^2\,[\lambda_k - 1 + \cot^2\delta]/2\,(\lambda_k - 1)$$

$k = 0, 1, \ldots$. When $\delta = 0$, $y(1) = 0$ or $J_n(\sqrt{\lambda_k - 1}) = 0$, and

$$\int_0^1 y^2\, x\, dx = y'(1)^2/2\,(\lambda_k - 1).$$

So

$$\| J_n(\sqrt{\lambda_k - 1}\, x) \|^2 = J_n(\sqrt{\lambda_k - 1})'/2(\lambda_k - 1).$$

If f is an arbitrary element of $L^2(0, 1; x)$, then it can be expanded in the L^2 sense as a series in terms of the eigenfunctions $\left\{ J_n(\sqrt{\lambda_k - 1}\, x) \right\}_{k=1}^{\infty}$:

$$f(x) = \sum_{k=1}^{\infty} C_k J_n(\sqrt{\lambda_k - 1}\, x)/\| J_n(\sqrt{\lambda_k - 1}\, x) \|_{L_2}$$

where

$$C_k = \int_0^1 f(x)\, J_n(\sqrt{\lambda_k - 1}\, x)\, x\, dx/\| J_n(\sqrt{\lambda_k - 1}\, x) \|_{L_2}.$$

It is well known that if f possesses in addition certain integrability and smoothness properties, then $f(x)$, or $\frac{1}{2}(f(x + 0) + f(x - 0))$, equals the series at each x.

If f is in $D_{(0,1]}$, then

$$Lf = \sum_{k=1}^{\infty} \lambda_k C_k J_n(\sqrt{\lambda_k - 1}\, x)/\| J_n(\sqrt{\lambda_k - 1}\, x)\|_{L_2}$$

in the L^2 sense.

Finally for all f in L^2 (0, 1; x)

$$(L-\lambda)^{-1} f = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} C_k J_n(\sqrt{\lambda_k - 1}\, x)/\| J_n(\sqrt{\lambda_k - 1}\, x)\|$$

in the L^2 sense provided, of course, $\lambda \neq \lambda_k$ for some k.

When $\lambda = 0$,

$$L^{-1} f = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} C_k J_n(\sqrt{\lambda_k - 1}\, x)/\| J_n(\sqrt{\lambda_k - 1}\, x)\|.$$

This shows via the Parseval equality that

$$\| L^{-1} f\|^2 = \sum_{k=1}^{\infty} C_k^2/\lambda_k^2 \leq \sum_{k=1}^{\infty} C_k^2 .$$

Since also by the Parseval equality,

$$\| f\|^2 = \sum_{k=1}^{\infty} C_k^2,$$

this establishes that L^{-1} is a bounded operator, bounded above by 1

$$\| L^{-1}\|_{L^2} \leq 1.$$

The extra term in the definition of l was inserted so that this would be true.

3. The Sobolev Spaces for (0, 1]

If l is multiplied by $x\bar{z}$, where z is "arbitrary", and the result is integrated from 0 to 1, we find

$$\int_0^1 (-(xy')' + [(n^2/x) + 1x]\, y)\bar{z}\, dx$$

$$= -(xy')\bar{z}\Big|_0^1 + \int_0^1 (xy'\bar{z}' + (n^2/x) y\bar{z} + xy\bar{z})\, dx.$$

Let us assume for the moment that the limit at $x = 0$ is 0. Then inserting the boundary term for $y(0)$ when $\delta \neq 0$, we have

$$\langle y, z \rangle_{H^1} = \int_0^1 (xy'\bar{z}' + (n^2/x)\, y\,\bar{z} + xy\bar{z})\, dx + \cot\delta\, y(1)\,\bar{z}(1)$$

which is a new inner product, generating a Sobolev space $H^1(0, 1; x; (n^2/x + x))$, which we denote merely by $H^1(0, 1)$.

When $\delta = 0$, the term at $x = 1$ cannot be evaluated. It is therefore necessary to place additional constraints on both y and z. First we assume that $y(1) = 0$ and $z(1) = 0$. We then define the $H^1(0, 1)$ inner product by

$$\langle y, z \rangle_{H^1} = \int_0^1 (xy'\bar{z} + (n^2/x)\, y\,\bar{z} + xy\,\bar{z})\, dx.$$

By itself, with no constraints at $x = 1$, this generates a Sobolev space. With the constraints we are in a subspace. It is easy to see that in the subspace, if $y_j \to y$ and $y_j(1) = 0$, then $y(1) = 0$ as well. The subspace is, therefore, closed, and is a Sobolev space itself. We denote it by $H^1(0, 1)$ as well.

Definition 3.1 — We denote by $\mathcal{D}_{(0,1]}$ those elements y in $H^1(0, 1)$ which satisfy

(1) y is absolutely continuous on all closed subsets of $(0, 1]$.

(2) (xy') is absolutely continuous on all closed subsets of $(0, 1]$.

(3) ly exists a.e. and is in $H^1(0, 1)$.

(4) If $n = 0$, y satisfies

$$\lim_{x \to 0} -xy'(x) = 0.$$

If $n > 0$, y satisfies

$$\lim_{x \to 0} x(nx^{n-1} y(x) - x^n y'(x)) = 0.$$

(5) For $0 \leq \delta \leq \pi/2$,

$$\cos\delta\, y(1) + \sin\delta\, y'(1) = 0.$$

We define the Bessel operator $\mathcal{L}$ by setting $\mathcal{L}y = ly$ for all y in $\mathcal{D}_{(0,1]}$.

Theorem 3.1 — For all y in $\mathcal{D}_{(0,1]}$ and z in $H^1(0, 1)$, $\lim_{x \to 0} -(xy')\bar{z} = 0$.

PROOF : When $n = 0$, $\lim_{x \to 0} -xy'(x) = 0$. The result then follows from the criterion developed in Krall and Race[4].

When $n \neq 0$, we note that $-xy'\bar{z}$ has a limit : Since y in $\mathcal{D}_{(0,1]}$ and z in $H^1(0, 1)$ imply ly and z are in $L^2(0, 1; x)$. So then the preliminary Dirichlet formula

$$\int_0^1 ly\bar{z}x\,dx = \int_0^1 \{-(xy')' + [n^2/x + x]y\}\,\bar{z}\,dx$$

$$= -xy'\bar{z}\Big|_0^1 + \int_0^1 (xy'\bar{z}' + [n^2/x + x]y\bar{z})\,dx$$

shows that the limit at 0 exists, since all the integrals converge.

Suppose $\lim_{x\to 0} -xy'\bar{z} = A \neq 0$. Then for x near 0, $|xy'||z| > |A|/2 > 0$. Since $x^{1/2}y'$ and $(n^2/x + x)^{1/2}z$ are in $L^2(0, 1)$, we find for some α near 0

$$\int_0^\alpha |x^{1/2}y'||(n^2/x + x)^{1/2}z|\,dx$$

$$= \int |xy'\bar{z}|\,(n^2/x + x)/x\,|^{1/2}\,dx$$

$$\geq |A|/2 \int_0^\alpha (n^2/x^2 + 1)\,dx = \infty.$$

This is a contradiction. Therefore $A = 0$.

Theorem 3.3 (The Dirichlet Formula) — For all y in $\mathcal{D}_{(0,1]}$ and z in $H^1(0, 1)$,

$$\int_0^1 ly\bar{z}x\,dx = \int_0^1 (xy'\bar{z}' + (n^2/x)y\bar{z} + xy\bar{z})\,dx + \cot\delta\, y(1)\bar{z}(1),$$

or, in inner product notation,

$$\langle Ly, z\rangle_{L^2} = \langle y, z\rangle_{H^1}.$$

PROOF : The limit at $x = 0$ is 0. At $x = 1$ we use the boundary condition satisfied by y.

Theorem 3.4 — $\mathcal{D}_{(0,1]}$ is dense in $H^1(0, 1)$.

PROOF : Suppose z is orthogonal to $\mathcal{D}_{(0,1]}$. Then

$$\langle y, z\rangle_{H^1} = \langle Ly, z\rangle_{L^2} = 0$$

for all y in $\mathcal{D}_{(0,1]}$. But Ly in L^2 can have z as an L^2 limit. Hence $\langle z, z\rangle_{L^2} = 0$ as well.

Thus $z = 0$ in L^2 and so $z = 0$ in $H^1(0, 1)$.

Theorem 3.5 — $\mathcal{L}$ is symmetric in $H^1(0, 1)$.

PROOF : Apply the Dirichlet formula with z replaced by $\mathcal{L}z = Lz$

$$\langle Ly, Lz\rangle_{L^2} = \langle y, \mathcal{L}z\rangle_{H^1}.$$

Reversing the roles of y and z and conjugating,

$$\langle Ly, Lz \rangle_{L^2} = \langle \mathcal{L}\, y, z \rangle_{H^1}$$

equate the rightsides.

Theorem 3.6 — The inverse $\mathcal{L}^{-1}$ exists and is bounded on $H^1(0, 1)$.

PROOF : In the Dirichlet formula let $y = z\ \mathcal{L}y = f,\ y = \mathcal{G}f$, where $\mathcal{G}$ is the Green's function operator in $L^2(0, 1; x)$. Then

$$\langle f, \mathcal{G} f \rangle_{L^2} = \langle \mathcal{G} f, \mathcal{G} f \rangle_{H^1}.$$

This implies that

$$\| \mathcal{G} f \|_{H^1}^2 \le \| f \|_{L^2}^2 \, \| \mathcal{G} f \|_{L^2}$$
$$\le 1 \, \| f \|_{L^2}^2 \ \le \| f \|_{H^1}^2.$$

If we divide $\|f\|^2$, take a square root and maximize over f, we see that the Green's operator, also generates $\mathcal{L}^{-1}$ and $\| \mathcal{L}^{-1} \| \le 1$ as well.

Theorem 3.7 — $\mathcal{L}$ is self-adjoint on $H^1(0, 1)$.

PROOF : Since $\mathcal{L}y = f$ can be solved for all f in $H^1(0, 1)$, $\mathcal{L}$ is maximally extended. Since it is symmetric, it is self-adjoint.

The $H^1(0,\ 1)$ *Spectral Resolution*

We are now faced with two related self-adjoint operators L and $\mathcal{L}$. The resolutions of the identity, L and L^{-1} are Bessel series. Those for $\mathcal{L}$ are at the moment unknown in form. Let Σf denote the L^2 Bessel series and $\int f$ denote the resolution of the identity associated with $\mathcal{L}$ in $H^1(0, 1)$.

Theorem 3.8 — The resolution of the identity associated with $\mathcal{L}$, $\int f$, is the same Bessel series, Σf, associated with L in L^2.

PROOF : According to the Dirichlet formula

$$\langle \int f, Lg \rangle_{L^2} = \langle \int f, g \rangle_{H^1}$$
$$= \langle f, g \rangle_{H^1} = \langle f, Lg \rangle_{L^2}.$$

Thus $\langle \int f - f,\ Lg \rangle_{L^2} = 0$ for all g in $\mathcal{D}_{(0,\ 1]}$. If we let Lg to approach $\int f - f$ in L^2, we see $\int f - f$ in the L^2 sense. But $\Sigma f = f$ in L^2, so $\int f = \Sigma f$ in L^2, a.e. and therefore in $H^1(0, 1)$. For all f in $H^1(0, 1)$, $f = \Sigma f$.

The Bessel series needs to be modified slightly in H^1. If $y_k = z_k = J_n \sqrt{(\lambda_k - 1}\, x)$, the Dirichlet formula shows

$$\langle y, y \rangle_{H^1} = \lambda_k \langle y_k, y_k \rangle_{L^2}.$$

Thus

$$\| J_n(\sqrt{\lambda_k - 1}\,x) \|_{H^1} = \sqrt{\lambda_k}\, \| J_n(\sqrt{\lambda_k - 1}\,x) \|_{L^2}.$$

Further, the Fourier coefficient

$$\begin{aligned}\langle f, J_n(\sqrt{\lambda_k - 1}\,x) \rangle_{L^2} &= (1/\lambda_k) \langle f, L\, J_n(\sqrt{\lambda_k - 1}\,x) \rangle_{L^2} \\ &= (1/\lambda_k) \langle f, J_n(\sqrt{\lambda_k - 1}\,x) \rangle_{H^1}.\end{aligned}$$

If these substitutions are made in the L^2 Bessel series, we find for an arbitrary f,

$$f(x) = \sum_{k=1}^{\infty} C_k J_n(\sqrt{\lambda_k - 1}\,x) / \| J_n(\sqrt{\lambda_k - 1}\,x) \|_{H^1}$$

where

$$C_k = \langle f, J_n(\sqrt{\lambda_k - 1}\,x) \rangle_{H^1}\, J_n(\sqrt{\lambda_k - 1}\,x) / \| J_n(\sqrt{\lambda_k - 1}\,x) \|^2_{H^1}.$$

The form is a bit different looking, but really the same.

Similar modifications are needed in the series for $\mathcal{L}f$ and $\mathcal{L}^{-1}f$.

The implication of these expansions in $H^1(0, 1)$ is, of course, that not only does the series converge in the L^2 sense for f, but the differentiated series also converges to f' in the L^2 sense.

4. The Interval $(0, \infty)$

Obviously much of what has been said needs only minor modification for the interval $(0, \infty)$. We shall only fill in the changes.

Definition 4.1 — We denote by $D_{(0, \infty)}$ those elements y in $L^2(0, \infty; x)$ which satisfy

(1) y is absolutely continuous on all closed subsets of $(0, \infty)$.

(2) (xy') is absolutely continuous on all closed subsets of $(0, \infty)$.

(3) ly exists a.e. and is in $L^2(0, \infty; x)$.

(4) If $n = 0$, y satisfies

$$\lim_{x \to 0} -xy'(x) = 0.$$

If $n > 0$, y satisfies

$$\lim_{x \to 0} (nx^{n-1} y(x) - x^n y'(x)) = 0.$$

We define the Bessel operator M by setting $My = ly$ for all y in $D_{(0, \infty)}$.

It is known[5] that at $x = \infty$, l is in the limit point case, and so no boundary condition is needed.

The spectrum of M is continuous on $(0, \infty)$. The resolution of an arbitrary element f in $L^2(0, \infty; x)$ is given via the Hankel transform[1, 6].

If $$F_n(s) = \int_0^\infty f(\xi) J_n(s\xi)\, \xi\, d\xi,$$

the Hankel transform of f, then

$$f(x) = \int_0^\infty F_n(s) J_n(sx)\, s\, ds,$$

the inverse transform.

If f is in $D_{(0,\infty)}$, then

$$Mf = \int_0^\infty (s^2 + 1) F_n(s) J_n(sx)\, s\, ds.$$

If f is again arbitrary,

$$(M - \lambda)^{-1} f = \int_0^\infty \frac{1}{s^2 + 1 - \lambda} F_n(s) J_n(sx)\, sds$$

provided λ is not in $[1, \infty)$.

When $\lambda = 0$,

$$M^{-1} f = \int_0^\infty \frac{1}{s^2 + 1} F_n(s) J_n(sx)\, sds.$$

Since $(s^2 + 1)^{-1} < 1$, we see that Parseval's equality

$$\int_0^\infty |f|^2 x\, dx = \int_0^\infty |F_n(s)|^2\, sds$$

yields $\| M^{-1} \|_{L^2} \leq 1.$

5. The Sobolev Space for $(0, \infty)$

Integration over $(0, \infty)$ yields the preliminary Dirichlet formula

$$\int_0^\infty (-(xy')' + [(n^2/x) + 1x]\, y)\, \bar{z}\, dx$$

$$= -(xy'\bar{z})\Big|_0^\infty + \int_0^1 (xy'\bar{z}' + (n^2/x)\, y\bar{z} + xy\bar{z})\, dx.$$

The limit at $x = 0$ disappears as before. At ∞, too, since $\int_{\alpha}^{\infty} (n^2/x^2 + 1)\, dx = \infty$, the same proof as in Theorem 3.1 shows $\lim_{x \to \infty} -xy'\bar{z} = 0$. Hence we define

$$\langle y, z \rangle_{H^1} = \int_0^\infty (xy'\bar{z}' + (n^2/x)\, y\bar{z} + xy\bar{z})\, dx.$$

The space generated by this inner product is a Sobolev space $H^1(0, \infty; x, (n^2/x + x))$, which again we shorten to $H^1(0, \infty)$.

Definition 5.1 — We denote by $\mathcal{D}_{(0,\infty)}$ those elements y in $H^1(0, \infty)$ which satisfy

(1) y is absolutely continuous on all closed subsets of $(0, \infty)$.

(2) (xy') is absolutely continuous on all closed subsets of $(0, \infty)$.

(3) ly exists a.e. and is in $L^2(0, \infty)0$.

(4) If $n = 0$, y satisfies

$$\lim_{x \to 0} -xy'(x) = 0.$$

If $n > 0$, y satisfies

$$\lim_{x \to 0} x(nx^{n-1} y(x) - x^n y'(x)) = 0.$$

We define the Bessel operator $\mathcal{M}$ by setting $\mathcal{M}y = ly$ for all y in $\mathcal{D}_{(0,\infty)}$.

Theorem 5.2 — For all y in $\mathcal{D}_{(0,\infty)}$ and $H^1(0, \infty)$,

$$\lim_{x \to 0} -(xy')\bar{z} = 0, \quad \lim_{x \to \infty} -(xy')\bar{z} = 0.$$

Proof : The case as $x \to 0$ is the same as in Theorem 3.2. Likewise, since $\int_{\alpha}^{\infty} (n^2/x^2 + 1)^{1/2}\, dx = \infty$, the same proof works for all $n \geq 0$.

Theorem 5.3 (The Dirichlet Formula) — For all y in $\mathcal{D}_{(0,\infty)}$ and z in $H^1(0, \infty)$,

$$\int_0^\infty ly\,\bar{z}\,x\, dx = \int_0^\infty (xy'\bar{z}' + (n^2/x)\, y\bar{z} + xy\bar{z})\, dx$$

or, in inner product notation,

$$\langle \mathcal{M}y, z \rangle_{L^2} = \langle y, z \rangle_{H^1}.$$

Theorem 5.4 — $\mathcal{D}_{(0,\infty)}$ is dense in $H^1(0, \infty)$.

Theorem 5.5 — $\mathcal{M}$ is symmetric in $H^1(0, \infty)$.

Theorem 5.6 — The inverse $\mathcal{M}^{-1}$ exists and is bounded on $H^1(0, \infty)$.

Theorem 5.7 — $\mathcal{M}$ is self-adjoint on $H^1(0, \infty)$.

The $H^1(0, \infty)$ *Spectral Resolution*

Just as before the resolutions of the identity, $\mathcal{M}$ and $\mathcal{M}^{-1}$ are unchanged.

Theorem 5.8 — The resolution of the identity associated with M, $\int f$, is the same as that given by the Hankel transform associated with $\mathcal{M}$ in L^2.

The proofs of all these theorems are unchanged from those in section 3.

6. Remarks

A moments reflection shows that while the L^2 theory can be shown to be valid for all $n > -1$, the H^1 theory fails for negative n, since J_n is not in the H^1 spaces. In fact, it is J_{-n} which is, for negative n, and so any such Sobolev extension must merely reproduce the theory for $n > 0$.

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