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GENERALIZED COMPLEMENTS OF A GRAPH

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Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 1$. For each set V_r in P, remove the edges of G inside V_r and add the edges $\overline{G}$, (the complement of G) joining the vertices V_r. The graph $G_k^P\ (i)$ thus obtained is called the $k(i)$-complement of G with respect to P. The graph G is $k(i)$-self complementary ($k(i)$-s.c) if $G_k^P\ (i) \cong G$ for some partition P of V of order k. Further, G is $k(i)$-co-self complementary ($k(i)$-co-s.c.) if $G_k^P\ (i) \cong \overline{G}$.

We determine (1) all $k(i)$-s.c trees for k = 2, 3, and (2) 2(i)-s.c. unicyclic graphs. Also, some necessary conditions for a tree/unicyclic graph to be $k(i)$-s.c. are obtained. We indicate how to obtain characterizations of all $k(i)$-co.s.c. trees, unicyclic graphs and forests from known results.

Key Words : Graphs; Complements; Trees; Unicyclic; Forests.

1. INTRODUCTION

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 1$. The k-complement G_k^P of G (with respect to P) is defined as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j, and add the edges which are not in G_p. The graph G is k-self complementary (k-s.c) with respect to P if $G_k^P \cong G$. For $2 \leq k \leq p$, characterizations of all k-self complementary trees, forests and unicyclic graphs are obtained in (1).

In this paper, we study another type of complement defined as follows :-

For each set V_r in the partition P, remove the edges of G inside V_r, and add the edges of $\overline{G}$ (the complement of G) joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)$-complement of G with respect to the partition P of V. The graph G is $k(i)$-self complementary ($k(i)$-s.c.) if $G_{k(i)}^P \cong G$ for some partition P of order k.

We observe that the Petersen graph is 2(i)-s.c. with respect to the partition $V_1 = \{1, 2, 3, 4, 5\}$, $V_2 = \{1', 2', 3', 4', 5'\}$. Also, it is 2-s.c. with respect to the partition

$$V_1 = \{1, 3, 4', 5'\},\ V_2 = \{2, 4, 5, 1', 2', 3'\}. \text{ (See Fig. 1).}$$

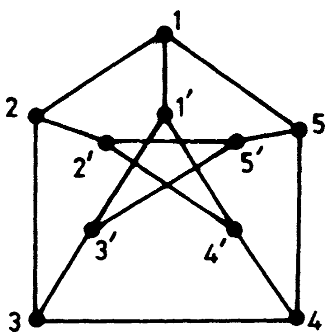

FIG. 1

If G is a graph of order P and $\overline{G}$ is the complement of G, we note that

i) $G_1^P \cong G$, and $G_{1(i)}^P \cong \overline{G}$, where $P = \{V\}$, and

ii) $G_p^P \cong \overline{G}$, and $G_{p(i)}^P \cong \overline{G}$, where P is the partition of V into singleton sets.

A $k(i)$-complement is nontrivial if $k < P$.

If $G_k^P \cong \overline{G}$, then G is k-co-self complementary (k-co-s.c.), and G is $k(i)$-co-self complementary ($k(i)$-co-s.c.) if $G_{k(i)}^P \cong \overline{G}$.

We determine (1) all $k(i)$-s.c. trees for $k = 2, 3$, and (2) all 2(i)-s.c. unicyclic graphs. Also, we obtain some necessary conditions for a tree/unicyclic graph to be $k(i)$-s.c. Further, we indicate how to obtain characterizations of all $k(i)$-co-s.c. trees, unicyclic graphs and forests from known results.

2. ELEMENTARY RESULTS

Proposition 1 — For any graph G,

$$i)\ \overline{G_k^P} \cong \overline{G}_k^P \qquad \text{and} \qquad ii)\ \overline{G_{k(i)}^P} \cong \overline{G}_{k(i)}^P$$

PROOF : *i*) Let u and v be two vertices in G. Then u and v are adjacent in $\overline{G_k^P}$

$\leftrightarrow$ u and v are nonadjacent in G_k^P.

$\leftrightarrow$ u and v are in the same set in P, and are nonadjacent in G, or they are in different sets in P, and are adjacent in G.

$\leftrightarrow$ u and v are in the same set in P, and are adjacent in $\overline{G}$, or they are in different sets in P, and are nonadjacent in $\overline{G}$.

$\leftrightarrow$ u and v are adjacent in $\overline{G}_k^P$.

This proves (i).

ii) u and v are adjacent in $\overline{G^P_{k(i)}}$

$\leftrightarrow$ they are nonadjacent in $G^P_{k(i)}$.

$\leftrightarrow$ u and v are in the same set in P and are adjacent in G, or they are in different sets in P and are nonadjacent in G.

$\leftrightarrow$ u and v are in the same set in P and are nonadjacent in $\overline{G}$, or they are in different sets in P and are adjacent in $\overline{G}$.

$\leftrightarrow$ u and v are adjacent in $\overline{G}^P_{k(i)}$.

This proves (*ii*).

As a consequence of Proposition 1, we have

Corollary 1.1 — For any graph G -

$$i)\ G^P_k \cong G \quad \leftrightarrow \quad \overline{G}^P_k \cong \overline{G}\,; \quad \text{and}$$

$$ii)\ G^P_{k(i)} \cong G \quad \leftrightarrow \quad \overline{G}^P_{k(i)} \cong \overline{G}.$$

In other words, G is k-s.c. if, and only if, $\overline{G}$ is so, and G is $k(i)$-s.c. if, and only if, $\overline{G}$ is so.

The k-complement and $k(i)$ complement of G are related as follows :-

Proposition 2 — *i)* $\overline{G^P_k} \cong G^P_{k(i)}$, and *ii)* $\overline{G^P_{k(i)}} \cong G^P_k$.

PROOF : *i)* Let u and v be two vertices in G. Then

u and v are adjacent in $\overline{G^P_k}$.

$\leftrightarrow$ they are in the same set of the partition P, and are nonadjacent in G, or they are in different sets in P, and adjacent in G.

$\leftrightarrow$ they are adjacent in $G^P_{k(i)}$.

From Propositions 1 and 2, we have the following:

Corollary 2.1 — For any graph G

$$i)\ G^P_k \cong \overline{G}^P_k \cong G^P_{k(i)}$$

and

$$ii)\ G^P_{k(i)} \cong \overline{G}^P_{k(i)} \cong G^P_k$$

By Proposition 2, we have

Corollary 2.2 — *i)* $G^P_k \cong G \quad \leftrightarrow \quad G^P_{k(i)} \cong \overline{G}$.

$$ii)\ G^P_{k(i)} \cong G \quad \leftrightarrow \quad G^P_k \cong \overline{G}.$$

In other words, G is k-s.c. if, and only if, G is $k(i)$-co.s.c., and G is $k(i)$-s.c. if, and only if, it is k-co-s.c.

3. $k(i)$-Co-Self Complementary Graphs

As usual, let P_r and C_r respectively denote a path and a cycle on r vertices.

Every complete bipartite graph $K_{m,n}$ is 2(i)-co-s.c. with respect to the partition $P = \{V_1, V_2\}$, where V_1 consists of two vertices corresponding to the end vertices of an edge and V_2, the other vertices. In particular, every star $K_{1,n}$, $n \geq 2$, is 2(i)-co-s.c. A double star is a tree with exactly two vertices of degree greater than one. Every double star is 2(i)-co-s.c., (V_1 is composed of the two vertices of degree greater than one), as are the cube Q_3 (V_1 is composed of the vertices on two diagonally opposite edges), the paths P_3 (a star), P_4 (a double star), P_5 (V_1 is the middle vertices $P_6(V_1)$ is composed of the two middle vertices and the two end vertices), P_7 (V_1 is composed of every other vertex), the cycle C_4 (V_1 is composed of two adjacent vertices), C_6 (V_1 is composed of two opposite vertices, and C_8 (V_1 is composed of every other vertex).

We now state a number of results on $k(i)$-co-s.c. graphs without proofs. These are immediate consequences of Corollary 2.2 and the results in [1] as indicated.

Proposition 3 — (Cf. Proposition 1 of [1]). If a (P, q)-graph G is $k(i)$-co-s.c., then

i) G has a vertex of degree at least $\dfrac{p(k-1)}{2k}$, and

ii) $\dfrac{(k-1)(2p-k)}{2k} \leq q \leq \dfrac{2p(p-k)+k(k-1)}{4}$.

Proposition 4 — (Cf. Corollary 1.1. of [1]) The following statements are true :-

i) A 4(i)-co-s.c. tree has order at most four.

ii) There are no $k(i)$-co-s.c. trees for $k \geq 5$.

iii) A forest with at least two components is not $k(i)$-co-s.c. for $k \geq 3$.

iv) A connected 4(i)-co-s.c. unicyclic graph has order at most six.

v) A connected 5(i)-co-s.c. unicyclic graph has order five.

vi) There do not exist connected $k(i)$-co-s.c. unicyclic graphs for $k \geq 6$.

3.1. *Characterizations of k(i)-co-s.c. Trees*

As mentioned above, using Corollary 2.2, characterizations of $k(i)$-co-s.c. trees can be easily deduced from the characterizations of k-s.c. trees obtained in [1]. We simply state the results without proofs.

Proposition 5 — (Cf. Theorem 3 of [1]) A tree of order P is 2(i)-co-s.c. if, and only if, one of the following holds:-

i) $P = 7$ and the tree is either P_7 or it consists of a path $v_1v_2v_3v_4v_5v_6$ together with a pendent edge v_4v_7.

ii) The vertex set of T can be partitioned into two sets V_1 and V_2 such that one of (a), (b) and (c) is true.

(a) $p \geq 5, |V_1| = 1$ and the subgraph $\langle V_2 \rangle$ has exactly $(P - 1)/2$ components.

(*b*) V_1 consists of exactly two nonadjacent vertices and exactly one component of $\langle V_2 \rangle$ is K_2, and all others, of which there is at least one, are K_1's.

(*c*) $V_2 = \{u, v\}$ where uv is an edge and V_1 is independent.

Corollary 5.1 — (*Cf.* Corollary 3.1 of [1]). Any $2(i)$-co-s.c. tree has diameter at most six.

Let T_1 and T_2 be two trees obtained from a path $P_5 : v_1v_2v_3v_4v_5$ as follows:

T_1 is P_5 plus two pendent edges v_3v_6 and v_3v_7; and

T_2 is P_5 plus two pendent edges v_2v_6 and v_4v_7.

Corollary 5.2 — (*Cf.* Corollary 3.2 of [1]) All trees on P vertices, $3 \le p \le 7$, except T_1 and T_2 are $2(i)$-co-s.c.

In a similar way, one can deduce characterizations of $k(i)$-co-s.c. forests and unicyclic graphs from the known results. The reader is referred to [1] for details.

4. $k(i)$-Self Complementary Trees

We first obtain some necessary conditions for a tree to be $k(i)$-s.c.

Proposition 6 — Suppose $P = \{V_1, V_2, \ldots, V_k\}$ is a partition of the vertex set $V(T)$ of a tree T, and $G_r = \langle V_r \rangle$ be the subgraph induced by V_r, $1 \le r \le k$. If $T^P_{k(i)} \cong T$, then -

i) $\beta_0(G_r) \le 2$ for $1 \le r \le k$, where $\beta_0(G_r)$ is the independence number of G_r;

ii) each set V_r in P contains at most four vertices; and

iii) each G_r is exactly one of the following graphs:

(*a*) rK_1, $1 \le r \le 2$, (*b*) K_2, (*c*) $K_1 \cup K_2$, (*d*) $K_{1,2}$, (*e*) P_4.

Proof : *i*) If $\beta_0(G_r) \ge 3$ for some G_r, then $T^P_{k(i)}$ contains a cycle, which is not true.

ii) This follows from (*i*), since otherwise $\beta_0(G_r) \ge 3$ for some G_r.

iii) By (*i*) and (*ii*), each G_r has at most four vertices with $\beta_0(G_r) \le 2$. If a G_r has at most three vertices, clearly, G_r must be one of the graphs mentioned in (*a*)-(*d*). If a G_r has four vertices, then it must have exactly three edges, since otherwise $\overline{G}_r$ has a cycle. Also $G_r \ne K_{1,3}$ by (*i*). This implies $G_r \cong P_4$.

Corollary 6.1 — If a tree T of order P is $k(i)$-s.c., then $k \le p \le 4k$.

The following result gives another necessary condition for a tree to be $k(i)$-s.c.

Proposition 7 — Let T be a $k(i)$-s.c. tree for some $k \ge 2$, with respect to a partition $P = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V.

Let v be a vertex of maximum degree in T, and f be an isomorphism of T onto $T^P_{k(i)}$. Then

deg $v \le k$ if $f(v) = v$; and

$$\deg\ v \leq \frac{k+4}{2} \text{ if } f(v) \neq v.$$

PROOF : Let $v \in V_1$. Then by Proposition 6, v has at most two neighbours in V_1. We now consider various cases.

Case 1 — $f(v) = v$.

If v has two neighbours in V_1, then by Proposition 6, the subgraph $\langle V_1 \rangle$ is either P_4 or $K_{1,2}$, and the degree of v in $T^P_{k(i)}$ is less than that in T, a contradiction. Hence, v has at most one neighbour in V_1. Also, v can have at most one neighbour in each of the other sets in P, for otherwise there will be a cycle in $T^P_{k(i)}$. Thus $\deg\ v \leq k$.

Case 2 — $f(v) = u \neq v$.

Subcase 2.1 — $u \in V_1$. In this case there exists at most one set other than V_1 containing neighbours of both u and v. Each other set has at most one neighbour of either v or u. This implies that $\deg\ v \leq 2 + 1 + \frac{k-2}{2} = \frac{k+4}{2}$.

Subcase 2.2 — $u \in V_2$.

Subcase 2.2.1 *(a)* — v is adjacent to u in T.

In this situation there is no set containing neighbours of both u and v.

Hence, $$\deg\ v \leq 2 + 1 + \frac{k-2}{2} = \frac{k+2}{2}.$$

Subcase 2.2.1 *(b)* — v is adjacent to a vertex different from u in V_2. Now, if u has a neighbour in V_1, then as in Subcase 2.2.1(*a*), $\deg\ v \leq \frac{k+4}{2}$. Otherwise, there exists at most one set $V_j, j \neq 1,2$ in P having neighbours of both u and v. Also, if v has two neighbours in V_1, then u is adjacent to a vertex in a set V_r which has no neighbours of v, since deg u in $T^P_{k(i)}$ must be equal to deg v in T. Hence $\deg\ v \leq 2 + 1 + 1 + \frac{k-4}{2} = \frac{k+4}{2}$.

Subcase 2.2.2. — v has no neighbours in V_2.

In this case there may exist at most two sets different from V_1 and V_2 having neighbours of both u and v. Hence,

$$\deg\ v \leq 2 + 1 + 1 + \frac{k-4}{2} = \frac{k+4}{2}.$$

Thus in all cases, $\deg\ v \leq \frac{k+4}{2}$. □

Corollary 7.1 — If a tree T is k-co-s.c. for some $k \geq 3$, then $\Delta(T) \leq k$.

We now characterize 2(*i*)-s.c. trees. In fact, one can find them.

Proposition 8 — There are exactly five trees which are 2(*i*) -s.c.

PROOF : Let T be a $2(i)$-s.c. tree of order P with respect to the partition $P = \{V_1, V_2\}$ of the vertex set V of T. Let G_r be the subgraph of T induced by V_r, $r = 1, 2$. By Corollary 6.1, $p \leq 8$. We consider various cases.

Case 1 — $P = 8$.

In this case it follows by Proposition 6 that every partition $P = \{V_1, V_2\}$ of V is such that $|V_1| = |V_2| = 4$. Again, by Proposition 6, $G_1 = G_2 = P_4$. The only tree that is $2(i)$-s.c. with respect to such a partition is the one in which a vertex of degree 2 in G_1 is adjacent to a vertex of degree 1 in G_2 as shown in Fig. 2.

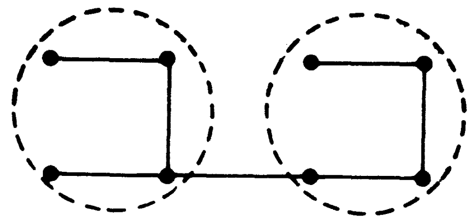

FIG. 2

Thus, there is exactly one tree on eight vertices which is $2(i)$-s.c.

Case 2 — $P = 7$.

In this case, it follows by Proposition 6 that $|V_1| = 4$ and $|V_2| = 3$, $G_1 = P_4$ and G_2 is either $K_1 \cup K_2$ or $K_{1,2}$. Both choices of G_2 yield a $T^P_{2(i)}$ in which the number of edges is different from that in T. This proves that no tree on seven vertices is $2(i)$-s.c.

Case 3 — $P = 6$.

As above we observe that either $|V_1| = 4$ and $|V_2| = 2$, or $|V_1| = |V_2| = 3$. In the first case, $G_1 = P_4$ and G_2 is either K_2 or $2K_1$. Both the choices of G_2 yield a $T^P_{2(i)}$ with different number of edges than T. If $|V_1| = |V_2| = 3$, then $G_1 = K_{1,2}$ and $G_2 = K_1 \cup K_2$ or vice-versa in order that both T and $T^P_{2(i)}$ have the same number of edges. There are exactly two trees on six vertices which are $2(i)$-s.c. with respect to such a partition. (See Fig. 3)

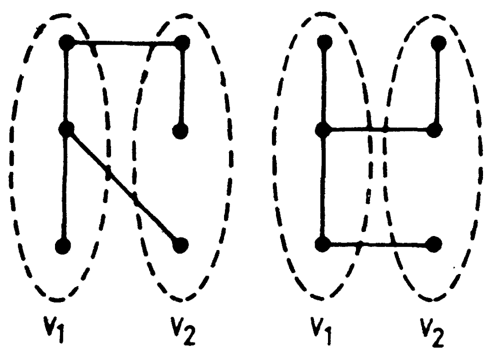

FIG. 3

Case 4 — $P = 5$.

In this case either $|V_1| = 4$ and $|V_2| = 1$ or $|V_1| = 3$ and $|V_2| = 2$. In the former case $G_1 = P_4$, and $G_2 = K_1$, and no tree is 2(i)-s.c. with respect to such a partition. In the latter case, $G_1 = K_{1,2}$ or $K_1 \cup K_2$ and $G_2 = 2K_1$ or K_2. The possible choices such that both T and $T^P_{2(i)}$ have the same number of edges are $G_1 = K_{1,2}$ $G_2 = 2K_1$, and $G_1 = K_1 \cup K_2$ and $G_2 = K_2$. One can easily verify that no tree obtained by joining a vertex of G_1 to a vertex of G_2 is 2(i)-s.c. Thus, no tree on five vertices is 2(i)-s.c.

Case 5 — $2 \leq p \leq 4$.

Out of the four trees in this case, only P_4 and K_2 are 2(i)-s.c. This proves the Proposition.

From Corollary 2.2, we have

Corollary 8.1 — There are exactly five trees which are 2-co-s.c., and these are the trees which are 2(i)-s.c.

We now characterize 3(i)-s.c. trees.

Proposition 9 — There are exactly seven 3(i)-s.c. trees.

PROOF : Let T be a 3(i)-s.c. tree of order P with respect to the partition $\{V_1, V_2, V_3\}$ of $V(T)$. Then by Proposition 6 $3 \leq p \leq 12$. Suppose $G = \langle V_r \rangle$, $1 \leq r \leq 3$. We consider various cases. In the proof we only consider partitions such that

(I) Both T and $T^P_{k(i)}$ have the same number of edges.

Case 1 — $P = 12$. By Proposition 6, $G_r = P_4$, $r = 1, 2, 3$, and one can verify that no tree with this partition is 3(i)-s.c.

Case 2 — $P = 11$. Again by Proposition 6, $G_1 = G_2 = P_4$, and G_3 is either $K_{1,2}$ or $K_2 \cup K_1$. There is no 3(i)-s.c. tree with this partition.

Case 3 — $P = 10$. As above we find that the only possible partitions are as follows :

$$G_1 = P_4, G_2 = K_{1,2}, G_3 = K_2 \cup K_1, \text{ or}$$

$$G_1 = P_4, G_2 = P_4, G_3 = K_2 \text{ or } \overline{K}_2$$

With these partitions, we find that the tree T is not 3(i)-s.c.

Case 4 — $P = 9$. In this case the only possible partitions are

$$G_1 = G_2 = P_4, \quad G_3 = K_1,$$

$$G_1 = P_4, \quad G_2 = K_{1,2}, \quad G_3 = \overline{K}_2$$

$$G_1 = G_2 = G_3 = K_{1,2} \quad \text{or} \quad \overline{K}_{1,2}$$

The first one yields a 3(i)-s.c. tree as follows:

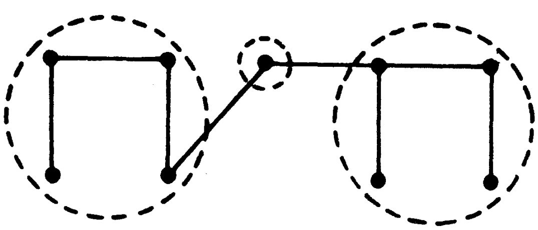

FIG. 4

The other partitions do not give 3(i)-s.c. trees.

Case 5 — P = 8. Among all possible partitions, we find that the only partition where $G_1 = P_4$, $G_2 = K_2$ and $G_3 = \overline{K}_2$ yield a 3(i)-s.c. tree as follows:

Case 6 — P = 7. In this case, the only partition which yields a 3(i)-s.c. tree is $G_1 = P_3$, $G_2 = K_2 \cup K_1$, $G_3 = K_1$.

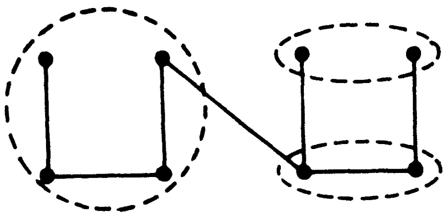

FIG. 5

There are exactly three trees which are 3(i)-s.c. with respect to such a partition, and they are as follows :-

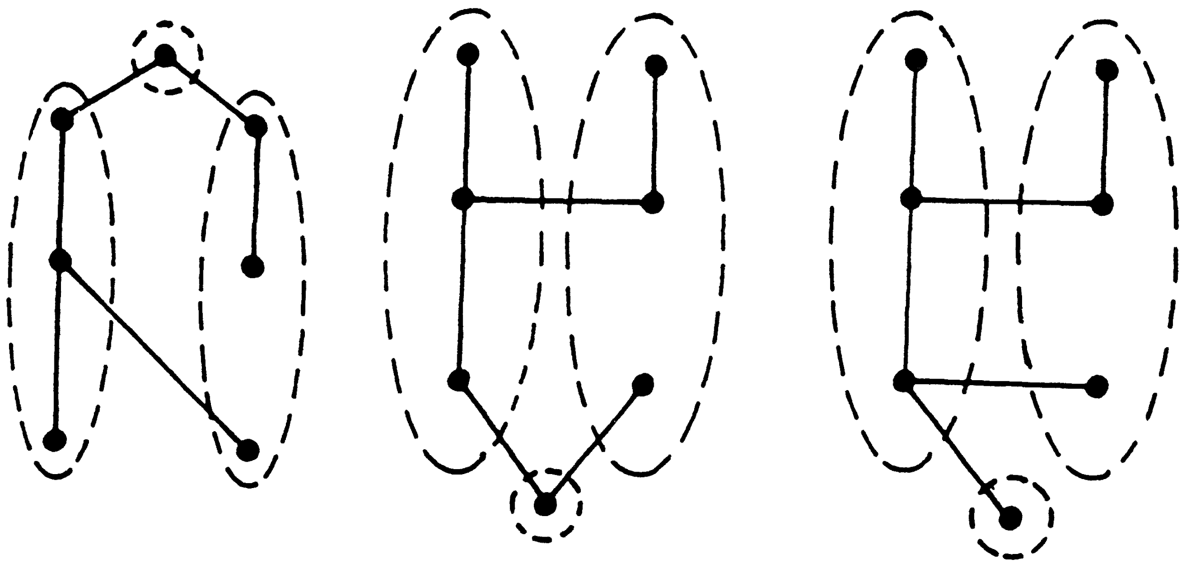

FIG. 6

Case 7 — P = 6. In this case, the only possible partitions are

$G_1 = P_4,\ G_2 = G_3 = K_1$

$G_1 = K_{1,2}, G_2 = \overline{K}_2, G_3 = K_1.$

There are no 3(i)-s.c. trees with such partitions.

Case 8 — P = 5. In this case there is exactly one tree namely, P_5 which is 3(i)-s.c. with respect to the partition $G_1 = K_2, G_2 = K_1, G_3 = \overline{K}_2$.

Case 9 — $3 \leq p \leq 4$. There is exactly one tree, namely, P_3 which is 3(i)-s.c. trivially,

By Corollary 2.2, we have □

Corollary 9.1 — There are exactly seven trees which are 3-co-s.c.

In general, characterization of $k(i)$-s.c. trees for $k \geq 4$ appears to be difficult. We now state some open problems.

Problem 1 : Characterize $k(i)$-s.c. trees for $k \geq 4$.

Problem 2 : Characterize trees which have nontrivial $k(i)$-complement which are also trees.

Problem 3 : Characterize graphs which are $k(i)$-complements of i) trees, and ii) cycles.

5. k-(i)-Self Complementary Unicyclic Graphs

We consider only connected unicyclic graphs here.

First we give some necessary conditions for a unicyclic graph to be $k(i)$-s.c., and then determine all 2(i)-s.c. unicyclicgraphs.

Proposition 10 — Let G be a unicyclic graph which is $k(i)$-s.c. with respect to the partition $P = \{V_1, V_2, \ldots V_k\}$ of $V(G)$. Then

i) $|V_r| \leq 5$ for $1 \leq r \leq k$, and

ii) for at most one V_r, $|V_r| = 5$, and in this case the subgraph $\langle V_r \rangle$ is a self complementary graph on five vertices.

Proof : If a set has six vertices, or two sets in P have at least five vertices each, then $G^P_{k(i)}$ has at least ~~two~~ cycles, a contradiction. Now, suppose $|V_1| = 5$. Then the subgraph $\langle V_1 \rangle$ should have exactly five edges, for otherwise $G^P_{k(i)}$ will have more than one cycle. Also, the subgraph $\langle V_1 \rangle$ must be a self complementary graph on five vertices, for otherwise, the lengths of the cycles in G and $G^P_{k(i)}$ will not be the same. □

Corollary 10.1 — If G is a $k(i)$-s.c. unicyclic graph of order P then, $k \leq p \leq 4k + 1$.

Proposition 11 — Let G be a unicyclic graph $P = \{V_1, V_2, \ldots, V_k\}$ $k \geq 3$ be a partition of the vertex set $V(G)$ of G, and $G \cong G^P_{k(i)}$. Then $\Delta(G) \leq k + 2$, where $\Delta(G)$ is the maximum degree of

G. Further, if for any vertex v of maximum degree in G and any isomorphism f of G onto $G^P_{k(i)}$, $f(v) \neq v$, then $\Delta(G) \leq \frac{k+6}{2}$.

PROOF : First we make two observations :-

1. A vertex in a self complementary graph on five vertices has degree at most three. This together with Proposition 10 implies that a vertex in any V_j, $1 \leq j \leq k$, has at most three neighbours in V_j.
2. A vertex v in any V_j, $1 \leq j \leq k$, can be adjacent to at most two vertices in any V_s, $s \neq j$. For otherwise, $G^P_{k(i)}$ will have more than two cycles.

Let v be a vertex of maximum degree in G, and $v \in V_1$. We consider various cases.

Case 1 — $f(v) = v$.

Since v has the same degree both in G and $G^P_{k(i)}$, we have

(A) : the number of neighbours of v in V_1 both in G and $G^P_{k(i)}$ must be equal.

Since $|V_1| \leq 5$, clearly (A) implies that in G, v cannot have three neighbours in V_1. We now consider various subcases.

Subcase 1.1 — In G, v has no neighbours in V_1.

It follows from (A) that $|V_1| = 1$, and there can be at most two other sets in P, say V_2, V_3 in each of which v may have two neighbours. Also, v may have at most one neighbour in each of the other sets. Hence, deg $v \leq 0 + 2 + 2 + k - 3 = k + 1$.

Subcase 1.2 — v has exactly one neighbour in V_1.

As in Subcase 1.1, v may have two neighbours in each of the sets V_2, V_3, and at most one neighbour in each of the remaining sets. So deg $v \leq 1 + 2 + 2 + k - 3 = k + 2$.

Subcase 1.3 — v has exactly two neighbours in V_1.

In this case, Proposition 10 and the condition (A) imply that the subgraph $\langle V_1 \rangle$ of G should be a self complementary graph on five vertices. So, v can have at most one neighbour in each of the other sets, and deg $v \leq 2 + k - 1 = k + 1$.

Case 2 — $f(v) = u \neq v$.

Subcase 2.1 (B) — v has three neighbours in V_1.

There may exist at most one set V_j, $2 \leq j \leq k$, say V_2 in which v has two neighbours. (For otherwise $G^P_{k(i)}$ will have more than one cycle).

Subcase 2.1.1 — v has two neighbours in V_2.

In this case it follows by (B) that

i) v belongs to a triangle in G, and since $f(v) = u$, u should also belong to a triangle in $G^P_{k(i)}$, and

ii) u belongs to either V_1 or V_2. (See Fig. 7)

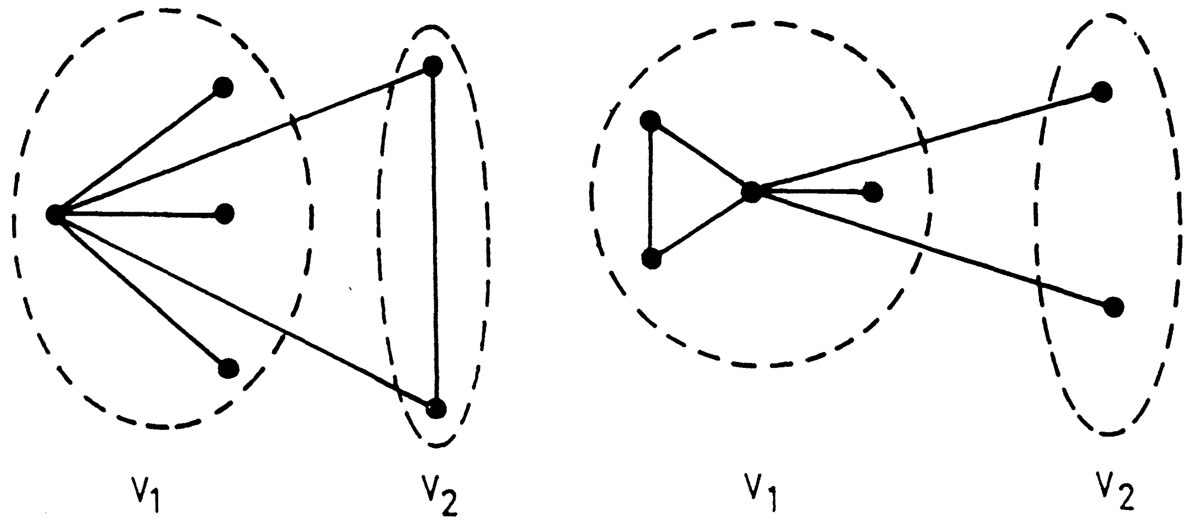

FIG. 7

u belongs to either V_1 or V_2. (See Fig. 7)

Subcase 2.1.1 (*a*) $u \in V_1$

If u has a neighbour in V_2, then both u and v cannot have neighbours in any V_j, $3 \leq j \leq k$. Also, u has exactly one neighbour in two sets in which v has no neighbours, since deg v in G is equal to deg u in $G^P_{k(i)}$. The remaining sets in P contain at most one neighbour of either u or v. Hence, deg $v \leq 3 + 2 + \frac{k-4}{2} = \frac{k+6}{2}$.

On the other hand, if u has no neighbours in V_2, then there exists at most one set V_j, $3 \leq j \leq k$, containing neighbours of both u and v. As above, we find that u has exactly one neighbour in three remaining sets in P in which v has no neighbour. The other sets in P contain a neighbour of at most one of u or v. Hence, deg $v \leq 3 + 2 + 1 + \frac{k-6}{2} = \frac{k+6}{2}$.

Subcase 2.1.1. (*b*). $u \in V_2$.

In this case both u and v cannot have neighbours in any V_j, $3 \leq j \leq k$. (For otherwise, $G^P_{k(i)}$, will have more than one cycle). Since deg v in G is equal to deg u in $G^P_{k(i)}$, it follows by (B) that if u has r neighbours (one each) in some r sets in the collection $P' = \{V_3, V_4, \ldots, V_k\}$, then v has at most $(r - 2)$ neighbours (one each) in some other $(r - 2)$ sets in P'. This implies that in G, deg $v \leq 3 + 2 + \frac{k-4}{2} = \frac{k+6}{2}$.

Subcase 2.1.2 — v has at most one neighbour in each set V_j, $2 \leq j \leq k$. In this case, both v and u can have neighbours in at most one set V_j, $2 \leq j \leq k$. Hence, as in Subcase 2.1.1., we have

$$deg\ v \leq 3+1+\frac{k-2}{2}=\frac{k+6}{2}$$

Subcase 2.2 — V has at most two neighbours in V_1.

In this case, there exist at most two other sets, say V_2, V_3 in each of which v may have two neighbours. In this case also, by similar arguments as above, one can show that deg $v \leq \frac{k+6}{2}$. □

We now find all $2(i)$-s.c. unicyclic graphs

Proposition 12 — There are exactly eighteen $2(i)$-s.c. unicyclic graphs.

PROOF : Let G be a $2(i)$-s.c. unicyclic graph of order P with respect to the partition $P = \{V_1, V_2\}$ of $V(G)$. By Corollary 10.1, $p \leq 9$. We consider various cases, and in each case, we consider partitions satisfying the condition

(D) : the number of edges in G and $G^P_{k(i)}$ are equal.

Case 1 — P = 9.

The only possible partition of $V(G)$ is such that $|V_1| = 5$ and $|V_2| = 4$. There is no $2(i)$-s.c. unicyclic graph with respect to such a partition.

Case 2 — P = 8.

The only possible partitions are such that $|V_1| = |V_2| = 4$.

Subcase 2.1 — Each subgraph $\langle V_1\rangle$ and $\langle V_2\rangle$ has exactly three edges. There are exactly four unicyclic graphs $G_1 - G_4$ as in Fig. 8 which are $2(i)$-s.c. with respect to such partitions.

Subcase 2.2 — The subgraph $\langle V_1\rangle$ has four edges and the subgraph $\langle V_2\rangle$ has two edges.

In this case, there are exactly five $2(i)$-s.c. graphs $G_5 - G_9$ as shown in Fig. 8.

Case 3 — P = 7. There are no partitions satisfying the condition (D).

Case 4 — P = 6.

There exist two types of partitions satisfying the condition (D),

namely $|V_1| = 5, |V_2| = 1$, and $|V_1| = |V_2| = 3$.

Subcase 4.1 — $|V_1| = 5, |V_2| = 1$.

There are exactly two $2(i)$s.c. unicyclic graphs G_{10} and G_{11} with this partition as shown in Fig. 8.

Subcase 4.2 — $|V_1| = |V_2| = 3$.

In this case there are exactly five graphs $G_{12} - G_{16}$ as in Fig. 8.

Case 5 — P = 5. There are two possibilities for the sets as follows :

$|V_1| = 4, |V_2| = 1$ and $|V_1| = 3,\ |V_2| = 2$.

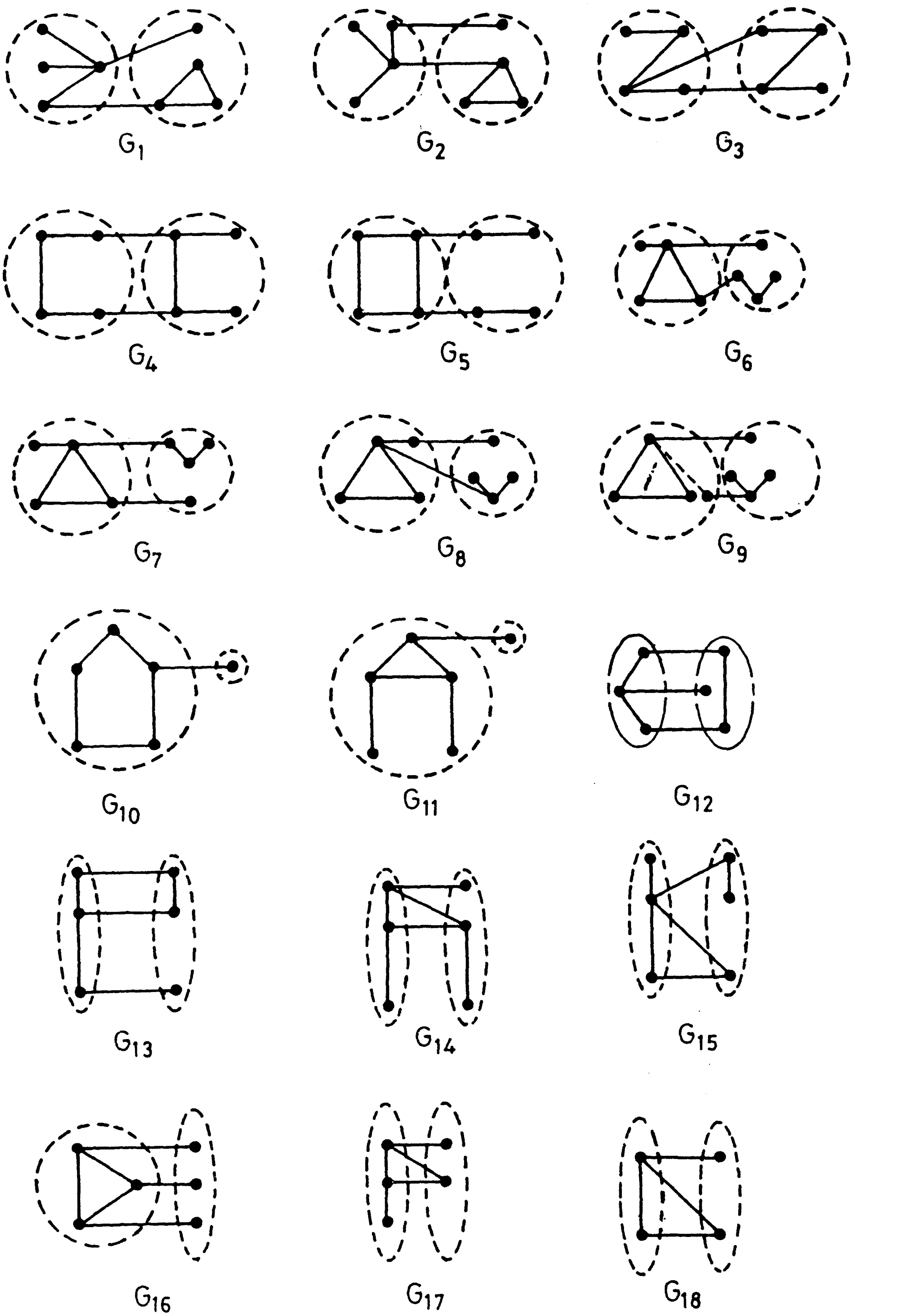

FIG. 8

Out of these only the latter gives a 2(i)-s.c. graph G_{17} as in Fig. 8.

Case 6 — $p \leq 4$. In this case G_{18} is the only graph which is 2(i)-s.c. □

Corollary 12.1 — There are exactly eighteen 2-co-s.c. unicyclic graphs namely, those in Fig. 8.

REFERENCE

1. E. Sampathkumar and L. Pushpa Latha, Complement of a graph- A Generalization, *Graphs and Combinatorics* (*to appear*)