Indian Journal of History of Science, 47.4 (2012) 771–832

BIRTH OF CALCULUS WITH SPECIAL REFERENCE TO YUKTIBHĀṢĀ

P. P. Divakaran*

(Received 7 February 2011)

Continuing a critical reading of the text *Yuktibhāṣā* of Jyeṣṭhadeva, two closely related questions are addressed: i) do the infinitesimal methods used in deriving the power series for trigonometric functions as well as the surface area and volume of the sphere mark the invention of the discipline of calculus? and ii) what are the sources of the ideas and techniques that culminated in these results? A careful analysis of the text and comparison with early European calculus lead to the conclusion that the critical concept, that of local linearisation, is common to both and that the answer to the first question is an unambiguous yes. The roots of this breakthrough, in particular the recognition of the need for 'infinitesimalisation' whenever the rule of three fails to hold, go back to the material in *Āryabhaṭīya* dealing with the sine table. Related issues examined include the misunderstanding of the mathematics of the sine table during the nine hundred years separating Mādhava from Āryabhaṭa and the contrasting attitude to abstraction and generalisation in the work described in *Yuktibhāṣā* and in the later European approach to similar problems.

Key words and phrases: Nila school; Power series; Local linearisation; Fundamental theorem of calculus; Decimal numbers; *Āryabhaṭīya*; Sine table; Difference, differential and integral equations.

*NCBS-TIFR, Bangalore 560065; Email: *ppdppd@gmail.com*

Calculus or not?

For students of the work of the astronomer-mathematicians who lived in Kerala on the southwest coast of India during ca. 1400-1600 CE (the Nila school[1]),[2] the appearance of two recent publications is a significant event. One, Kim Plofker's panoramic survey of Indian mathematics over its known history and geography [1][3], has a detailed overview of most of their achievements and places them in their proper historical context. The other is K. V. Sarma's English translation of *Yuktibhāṣā* [4], a work which is, as I shall argue below, indispensable for an appreciation of the originality and power of the Nila school and hence is *the* key text for the purpose of the present article. That purpose is to try and establish, from a detailed look at the material in *Yuktibhāṣā*, that the most original and powerful of their achievements was the invention of a novel mathematical discipline that we now call calculus, though in the limited context of certain 'elementary' questions involving trigonometric functions. And what makes *Yuktibhāṣā* such an indispensable sourcebook is that its expansive text provides proofs of every relevant result cited (with two exceptions which are not central to our concerns here) as well as prefatory and explanatory passages on the context and motivation of these results. Such asides, largely absent in other texts of the Nila school as indeed they are from almost all of Indian mathematical writing, will turn out to be invaluable in making the case that the invention of calculus came as the innovative response from Mādhava, the founder of the school, to a need felt from as long back as Āryabhaṭa (499 CE).

As is well-known to historians of mathematics now, the most spectacular of the results of the Nila school are power series expansions of certain trigonometric functions, namely, the arctangent series:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$$

and its specialisation to $\theta = \pi/4$ (the basic π series):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots;$$

the sine series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots;$$

and the cosine series:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots .$$

Precisely these are among the first landmarks of late 17th century European calculus, as attested by their association with the names of Newton, Leibniz and Gregory. More than three centuries later, there still is no way of getting at these series, especially the π series, that does not rely indispensably on the central idea of infinitesimal calculus, that of the processes of local linearisation (differentiation) and its inverse (integration) as encompassed in the fundamental theorem of calculus. It would appear that this fact has not had much of an effect on the continuing debate in some circles on whether the Nila work was really calculus or only some 'form of calculus', or even a sort of 'precalculus'.[4] As though to compensate, others have suggested that a general idea of calculus was intuitively apprehended in India well before Mādhava came on the scene. By the end of this article, I hope to be able to convince the patient reader that these incompatible positions are both equally untenable. As much as the publication of the two books mentioned in the beginning, this risk of the possibility of a double misunderstanding of the achievements of the Nila school provides an additional motivation for a critical fresh look at the whole question.[5]

Less widely noted than the trigonometric power series are other sections in *Yuktibhāṣā* that describe a way of deriving the formulae for the surface area and volume of a sphere. These formulae have a long history, the first correct derivation going back, of course, to Archimedes and his beautiful method of exhaustion. In India, while Āryabhaṭa got them famously wrong, Bhāskara II (mid-12th century CE) employed a logically questionable semi-numerical method which is not truly infinitesimal but arrived at the correct formulae (for a reason which it will be instructive to look at later). Their treatment in *Yuktibhāṣā*, in contrast, displays a clear understanding of the new infinitesimal method of Mādhava. Indeed, once dressed up in symbols and today's style of presentation, the *Yuktibhāṣā* proofs are essentially the same as can be found in a textbook of elementary calculus, with the added bonus that it includes the actual working out of the (definite) integral of sine and cosine (by a method which will appear quite original in the modern classroom).

To assert that the writings of the Nila school record the birth of the discipline of calculus is to invite a whole host of questions. The first of them, inevitably a very general one, is: what exactly are the hallmarks of this discipline? Why do we consider Fermat's quadrature of the 'general parabola' ($y^m = ax^n$), for example, to be 'calculus', a direct antecedent of the calculus of Newton and Leibniz while Archimedes's method of exhaustion is generally regarded not to be? In India, correspondingly: why are Bhāskara II's formulae for the sphere not to be thought of as marking the advent of calculus? Or: what is the status of Mādhava's interpolation formulae for the sine and the cosine as a possible illustration of his own foundational principles of true calculus? The interpolation formula has already been looked at in this light in [7], with the conclusion that, even if iterated indefinitely, it is most certainly not an illustration of the method of calculus and is in fact wrong. The sources of Bhāskara's formulae are historically more interesting and is one of the questions that will be examined with some care below. The two examples are valuable as they provide excellent case studies of criteria that will separate what should be considered calculus and what not and, in that role, have figured in recent discussions – the interpolation formula is treated at length in [8] (useful also to trace the literature questioning the calculus credentials of the Nila work) while Mumford's recent review [9] of [1] looks at the Nila work as a whole as well as Bhāskara's formulae in the perspective of modern calculus.

The issue of what exactly one means by calculus, in the context of its European history, is an old and frequently addressed one to which little worthwhile can be added anew. Neverthelesss, given the somewhat different metaphysics of calculus (to use a phrase favoured by d'Alembert and Lazare Carnot among others) in its Indian avatar it is useful, and perhaps even obligatory, to revisit the question and frame the present discussion within its ambit. I will argue that the notion that captures the essence of calculus in both the Indian and the European approaches is most naturally and intuitively expressed geometrically: the local linearisation of a curve and its relationship to determining its length (rectification), in other words the fundamental theorem of calculus in its most elementary geometrical version. This is not a very original thing to say but it will help free the discussion from irrelevancies.

Especially noteworthy within this framework is that the notion of an infinitesimal, which caused such controversy in Europe already while Newton and Leibniz were alive, has no primacy in the Indian approach. What was central was its inverse notion, that of an infinite (more accurately, unboundedly large) quantity as exemplified by the infinitude of natural numbers: a (finite, geometric) quantity was 'infinitesimalised' by dividing it by a large positive integer which was then allowed to grow without bound. There are other important differences in the way the metaphysics of calculus played out in India and in Europe, discussed in section 8 below, but it will become clear nevertheless that they do not detract from the underlying conceptual and methodological unity of the two traditions: the truly innovative element in the Nila corpus *is* the invention of calculus rather than trigonometric novelties or even the recognition of infinite series as meaningful mathematical objects.

That in turn raises fresh questions. Why were these brilliant men content just with the examples listed above, without the urge to explore, as in Europe, the generality of the ideas and methods they had in hand? And, from a historiographic angle, why has there been a hesitancy among some historians of mathematics to acknowledge what is most deeply original in their work? There can be no final objective answers to such questions. All that is possible is to offer plausible suggestions, with due respect paid to the original texts, especially but not exclusively *Yuktibhāṣā*.

The present article is the second in a short series on some general themes in the evolution of mathematical thought in India, as seen from the vantage point of its last great phase and the most insightful text of that phase. As in the earlier paper [7], part of the aim is to trace the roots of the ideas that came to maturity in this final phase. As far as calculus is concerned, its geometric/trigonometric roots go back directly to certain specific verses from the *Āryabhaṭīya* and thence, more broadly, to the original Vedic (*Śulbasūtra*, ca. 8th century BCE) geometry of circles and right triangles. The other, 'infinitesimal', component comes, as noted earlier, from long familiarity with the infinitude of numbers as expressed in the decimal place-value system of enumeration, already mastered in the early Vedic period as is evident from the number names of the *Ṛgveda* (ca. 12th-11th centuries BCE [7,10,11]). In a more abstract direction, decimal numbers also served as a model for polynomials and power series ([6,11]) which figure so prominently in the Nila material on calculus as they did, later, for Newton ([7]).[6]

Also as in [7], the general method of presentation adopted here is to start with the relevant passages from *Yuktibhāṣā* and then to trace the roots of the key ideas back as far as we can go. The passages cited or quoted are identified by the section and subsection numbers in Sarma's book ([4]) though the actual quotations when given are in my own translation. In order to minimise the potential for misreading or overreading, I have tried to make the translations as literal as is consistent with the demands of English. When geometrical constructions are invoked, I shall fall back on the clear diagrams in the supplementary notes ([13]) to [4].[7] For this reason among others (and as the title is meant to convey), the reader will find it useful to have a copy of [4] to hand.

Given that recursive methods (most notably the method of successive refining called *saṃskāram* in *Yuktibhāṣā*), which were the focus of the earlier paper [7], form such an important part of the technical apparatus of the Nila calculus, it has turned out to be difficult to avoid a certain degree of overlap between the present paper and the earlier one. The introductory section of [7] serves, in particular, as an overview of calculus as well as of recursion.

Finally, in the interests of historical completeness, I have also thought it worthwhile to preface the main body of the article with a brief but critical update of whatever is reasonably securely known about the lives of the main protagonists.

Mathematicians in their villages

In the context of the little we know of how mathematics was transmitted over the generations and from region to region, the Nila school is quite exceptional in the continuity and longevity of its productive life. Every link in the line, teacher-to-pupil and occasionally familial, from Mādhava (born around the middle of the 14th century) to one of its later recognisably significant members (Acyuta Piṣāraṭi, died around 1620) is traceable with a fair degree of reliability ([15,16,1])[8]. Equally striking is the mutual proximity of the villages associated with most of the major figures, almost all of them on the northern bank of the Nila, close to its confluence with the ocean.

The Nila (also called Pērār in the past, nowadays more commonly known as the river Bhārata) is only just above 200 km long though it is the second longest river in the narrow strip of land that is Kerala. In its lower basin it is (used to be) a splendid river, very wide with extensive sand banks and reed beds. As far back as records and communal memory go, it has been the theatre in which the cultural, intellectual and, to an extent, the political history of Kerala played out, attaining great prominence during the reign of the Zamorin (Sāmūtiri) dynasty ruling from Kozhikode (Calicut). The Zamorins were not only the political overlords of the region during the period of our interest, but also drew religious and secular legitimacy from its many famous temples and from the rites of kingship celebrated on the banks of the river. They were also great patrons of scholarship, maintaing a royal academy in the capital and generally continuing a much earlier tradition of supporting centres of learning. It is tempting to suppose that such a centre existed in the Nila region, acting as a focus for the teaching and doing of mathematical and observational astronomy, but there is no recorded evidence for such a thing.[9]

The main 'mathematical villages' are on the right (north) bank of the Nila, extending from Tirunavaya (Tirunāvāya), situated right on the river, to Trikkandiyur (Tṛkkaṇṭiyūr), some 10 km to the north, with Triprangode (Tṛpṛaṅṅōd) and Alathiyur (Ālattiyūr)[10] to the west of this north-south axis. None of them is much more than about 5 km from the seashore, scene of much action during the 16th and 17th centuries as the Zamorins fought off the Portuguese at sea and on land. Two other villages, at some distance from this cluster, also deserve mention: Shukapuram (Śukapuram) to the south of the river and Trikkudaveli (Tṛkkuṭavēli) some 50 km upstream to the east, the natal villages of Citrabhānu and Śaṅkara (Vāriyar) respectively. It is to be noted that every one of these villages has a major temple. Temples served as educational, social and cultural centres in the past; in the social and cultural spheres they still do.

The following sketch of the lives of the main protagonists and their Nila affiliations is largely based on the picture drawn by K. K. Raja ([17]) and K. V. Sarma ([15]) from their unmatched knowledge of manuscripts, supplemented by the results of local enquiries. The two sources are not always congruent; some, a very few in fact, of the conclusions of Raja and Sarma are based on material which local knowledge and beliefs allow to be read slightly differently.

Of all the personalities of the Nila school, it is about Mādhava (Madhavan Emprantiri 'of Sangamagrama'), the founder of the school and inventor of calculus, that we know the least in regard to personal details. In fact, we know nothing apart from what we can tease out of the village name Sangamagrama ('the village at the confluence') and the subcaste epithet 'Emprantiri'. His dates (ca. 1350-1420?) are an educated guess, not likely to be far wrong ([17,1]). It is widely believed ([15] and many other references including [1]) that Sangamagrama is near Irinjalakkuda, approximately 50 km to the south of the Nila. The reason for the belief is, at best, tenuous. There are other places which have an equal or better claim to be Sangamagrama: i) a village named Kudalur (Kūṭalūr) whose literal translation gives the Sanskrit 'Sangamagrama', at the confluence of the river Kunthi with the Nila, about 15 km up the river from Tirunavaya and ii) Tirunavaya itself which used to be and sometimes still is referred to as Trimurtisangamam on account of the presence, on either bank of the river, of temples (predating the Nila period) dedicated to all three of the main Hindu deities. If only on the strength of proximity, it would be nice to be able to find real evidence for one or the other of these possible identifications.[11]

As for the name Emprantiri, in the early centuries of the second millennium, it was descriptive of brahmins who or whose immediate ancestors had come to Malabar (an old Arabic name for the Kerala coast, but nowadays reserved for its northern half) from farther north along the coast (the Tulu country) – most of the brahmins of Kerala (Namputiris) had migrated there in several waves, starting perhaps in the 7th-8th century CE, from what is now Maharashtra via the same Tulu coast. Maharashtra and Karnataka have many villages named after the confluence of rivers, with temples dedicated to the 'lord of the confluence'. Is this circumstance enough to suggest that Mādhava's family was a recent arrival from the country to the north and, more consequently, that he may thus have been instrumental in reestablishing the link beween mainstream Indian mathematics (e.g., Bhāskara II) and Kerala?

Some of the individuals who carried forward Mādhava's teachings – like Parameśvara, Mādhava's direct disciple and a great astronomer – have only incidental roles in the story of calculus. The central figures for us are Nīlakaṇṭha, Mādhava's great-grand pupil and the author of *Tantrasaṃgraha* and many other books; Jyeṣṭhadeva whose only (extant) piece of writing appears to be *Yuktibhāṣā*; and Śaṅkara (Vāriyar) who wrote *Yuktidīpikā* and *Kriyākramakarī* among other works. The last two were both disciples of Nīlakaṇṭha and both claimed that their own books *Yuktibhāṣā* and *Yuktidīpikā* were no more than commentaries on his *Tantrasaṃgraha*.

Nīlakaṇṭha's life is relatively well documented in his own writings as well as in his contemporaries' and disciples'. Born in Trikkandiyur in Kelallur house (*mana*) in 1444, he seems to have been closely associated with the temple in Alathiyur. A man of many parts and good in all of them – mathematician-astronomer, epistemologist, philosopher, expert in ritual, influential adviser to the politically powerful – he comes closest to being the conscience-keeper of the Nila school and has been its emblematic representative for the generations that followed. He lived at least till the 1520s.

The dates of Jyeṣṭhadeva are only roughly known. His apprenticeship under both Nīlakaṇṭha and Dāmodara (Nīlakaṇṭha's teacher and Parameśvara's son) would suggest that he was born some time in the last quarter of the 15th century. He is mentioned by both Śaṅkara and Acyuta with reverence; that and certain indirect inferences to be drawn from *Yuktibhāṣā* about its date (see the next section) would seem to indicate that he lived well into the second half of the 16th century[12]. As for where he came from, K. V. Sarma in a fine piece of detective work (Introduction (in English) to [18], building on a suggestion of Kunjunni Raja [17]), has linked a reference in mixed Malayalam and Sanskrit in an obscure manuscript that says, "Jyeṣṭhadeva, disciple of Dāmodara, is [a] Paṛaṅṅōṭṭu Nampūtiri, the same person who composed *Yuktibhāṣā*" to the common colophon at the end of each chapter of *Yuktidīpikā* which includes the phrase "the venerable twice-born (brahmin) living in Parakroḍa" (*parakroḍāvāsa dvijavara*). The conclusion, that Jyeṣṭhadeva belonged to Paṛaṅṅoḍ (the ending ...ṭṭu in the quote indicates the genitive or possibly the ablative case in Malayalam) and that Parakroḍa is its homophonic Sanskrit rendering, is credible.[13]

But what and where was this place? Sarma takes the view that it was the name of the ancestral house (*illam* or *mana*) or family to which Jyeṣṭhadeva belonged. That is possible but not required by either grammar or common usage. More reasonable and equally consistent with grammar and usage is the interpretation that "Paṛaṅṅōṭṭu Nampūtiri" means "the brahmin of (or from) Tṛpṛaṅṅod [village]";[14] *tiru* or *tṛ* (hallowed, sacred, auspicious) is a prefix often attached to the names of places with major temples (Tirunāvāya: the sacred spot of the Nāvā Mukunda temple). That will bring a pleasing unity, with every important temple of the region having a mathematical/astronomical connection. (It should be added that all the temples mentioned date from well before the period of our interest).

Śaṅkara also, though a prolific author, was reticent about his personal details. But there are quite a few bits of circumstantial evidence, ranging from his tribute to "the venerable brahmin" to the estimated dates of his two major books *Yuktidīpikā* and *Kriyākramakarī*, all well analysed by Sarma ([18]), establishing that he was perhaps a generation younger than Jyeṣṭhadeva. That is consistent with the dates assigned to Jyeṣṭhadeva here and with Plofker's conclusion ([1] citing Pingree) that Śaṅkara's productive period spanned the middle third of the 16th century, though it is slightly at odds with Sarma's dating of Jyeṣṭhadeva. Tṛkkuṭaveli, his native village according to Sarma ([18]), is on the Nila but about 50 km upstream, making it the most far-flung of the places associated with the core of the Nila group.

From the social-historical angle, the notable fact about Śaṅkara is his caste name Vāriyar, making it clear that he was not a Nampūtiri. He was thus the first known nonbrahmin astronomer-mathematician of the Nila school, indeed of Kerala and maybe even of all Hindu India. That fits in with the social practices of brahmins in Kerala, in particular their marriage customs. Some time around the 12th-13th centuries, they began following an extreme form of primogeniture in which only the eldest son could formally marry a brahmin woman and father brahmin children. Younger sons made perfectly open and legitimate but less 'official' alliances with non-Nampūtiri women; their offspring were not considered brahmins and carried caste apellations like Vāriyar, Piṣāraṭi, etc. They had access to temples but for the inner sanctum, and the portals of learning were open to them. With the effective birth rate of Namputiris thus brought down substantially below what was the norm, the proportion of brahmins fell over the centuries. It was perfectly natural then for non-brahmins having brahmin fathers like Śaṅkara and Acyuta to have made up for the shortfall and risen, in due course, to intellectual prominence[15].

The texts

Of whatever Mādhava himself might have written about his mathematical discoveries, nothing but a few fragmentary formulaic verses cited by his followers has survived. The loss is doubly regrettable. First of all, we are left ignorant of his immediate mathematical antecedents, a lack which is not adequately made up by the later writers who, meticulous though they are in the presentation of his work, are of no help in tracing the evolution of the specific ideas and techniques that they describe. Our only hope then is to prospect for whatever mathematical scraps (as distinct from the common mathematical heritage, e.g., the sine table of Āryabhaṭa) we can find in the writings of his direct predecessors. Such an enterprise is not very productive;[16] their surviving writings do not presage any of his results or deep ideas.

Secondly, what would one not give to come upon a miraculously preserved '*Mādhavīya*', to be able to read an account of Mādhava's achievements in Mādhava's own words? Was he aware, as Āryabhaṭa was and Newton and Leibniz in their time were, of the revolutionary transformation that he had brought about in mathematical thought? Would he have let posterity know, as Āryabhaṭa ceratinly did,[17] that he was? Most fascinatingly, how would he have acknowledged his debt to Āryabhaṭa's vision? – for there can be little doubt that one of the strands that Mādhava wove into the fabric of his calculus, infinitesimal geometry, leads directly back to Āryabhaṭa, owing little to the work of the intervening 900 years. There is, alas, no *Mādhavīya*,[18] so all we can try to do is to guess, but with discipline and rigour.

As though to make up for Mādhava's silence, most of his intellectual descendents turned out to be very articulate as far as the details of what he accomplished are concerned, filling the gap adequately each in his own way. They were also unanimous in giving him credit for all the novel ideas and results, including important auxiliary 'lemmas', (e.g., *jīveparaspara-nyāyam*, the addition theorem for the sine function), none more freely than Śaṅkara. The following summary is concerned only with those texts which matter in the story of calculus.

The *ur*text of the Nila school's new mathematics as well as the astronomical refinements it gave rise to is, by common consent, Nīlakaṇṭha's *Tantrasaṃgraha* ([18]) written in 1500. It is a relatively compact work of 431 two-line verses divided into 8 chapters of unequal length. The chapter headings make it clear that it was meant mainly as an astronomical compendium. The new mathematics of Mādhava is to be found mostly in chapter 2, the second longest. There are no proofs. Even outline justifications of the new results quoted are absent, which is not surprising since nothing less than a full *yukti* would have convincingly established their validity. Nīlakaṇṭha certainly could not have said, as he did about the theorem of the diagonal (Pythagoras' theorem), that they should be "self-evident to the intelligent" – there is nothing self-evident about the π series for example. There are also few explanations of the logic of what is being attempted and accomplished, especially in the purely mathematical parts. In short, *Tantrasaṃgraha* is not the kind of book that helps us get behind the hard facts and take a look at the 'metaphysics'.

Much (20 or 25 years) later, in the wisdom of his old age, Nīlakaṇṭha wrote a commentary, *Āryabhaṭīyabhāṣya*, on the three substantive chapters of *Āryabhaṭīya*. This is in extensive Sanskrit prose and full of insights that are found nowhere else – perhaps the most profound of Nīlakaṇṭha's works. Of special value to us are comments illuminating the chapter on mathematics (*Gaṇitapāda*), as they help in tracing the roots of Mādhava's calculus to Āryabhaṭa. Aside from the mathematical insights, these comments occasionally come in handy in the mundane business of dating texts. For instance, Nīlakaṇṭha's well known remark (conjecture?) on the irrationality of π and his turning of Āryabhaṭa's approximate treatment of the difference equation for the sine into an exact one, both of which are central to the concerns of *Yuktibhāṣā* and will be discussed later, find no echo in it, a work written by his own disciple following, self-professedly, his own *Tantrasaṃgraha*. A reasonable inference is that *Yuktibhāṣā* was written after *Tantrasaṃgraha* but *before Āryabhaṭīyabhāṣya*. Since Nīlakaṇṭha wrote two more books at least after *Āryabhaṭīyabhāṣya*, one of them a major work, we may conclude with some confidence either that he remained intellectually sharp well into his eighties or that *Yuktibhāṣā* itself is to be dated not much after 1520 (Sarma's preferred date is around 1530 [19]).

For the details of the reasoning (*yukti*) that *Tantrasaṃgraha* does not provide, we have to turn to the two much longer works of his disciples, *Yuktibhāṣā* and *Yuktidīpikā*, both of which start off by acknowledging their debt to *Tantrasaṃgraha*. Though the very first sentence of *Yuktibhāṣā* says: ". . . [I] begin by explaining all the mathematics useful in the motion of heavenly bodies following *Tantrasaṃgraha* . . .",[19] it is more an independent treatise than a canonically organised *bhāṣya* or *vyākhyā*. *Yuktidīpikā* also says (in its third verse) that it is written as a detailed analytic commentary (*vyākhyā*) on *Tantrasaṃgraha* and actually follows the appropriate format, with a verse or a group of verses taken up for detailed, sometimes very detailed, exposition. The thematic unity of the two books, inspired by their common source, is further underlined by *Yuktidīpikā*'s acknowledgement, in the chapter-ending verses mentioned earlier, of what it owes to Jyeṣṭhadeva: "Thus have I set out . . . the exposition that has been well-stated by the revered brahmin of Parakroḍa . . ." (Sarma's translation [18]).

Nevertheless, the two are often different in the relative importance given to the computations and propositions that make up the whole picture, a prime instance being their treatment of the interpolation formula of Mādhava. In fact the approach of Śaṅkara to the whole circle of ideas leading to the sine/cosine series is in a markedly different perspective from Jyeṣṭhadeva's. This has implications for the ongoing 'calculus or not?' debate. More importantly, it shines a light on the different perceptions the main personalities of the Nila school had of the the new mathematics of Mādhava they were gradually coming to terms with.

For a scientific text in metrical Sanskrit, *Yuktidīpikā* is exceptionally long – for instance, the 80 stanzas of the second chapter of *Tantrasaṃgraha* get 1102 stanzas of commentary from Śaṅkara, a veritable *tour de force* of mathematical versification. There are detailed demonstrations of the power series representations and the interpolation formulae are given a very elaborate treatment. This particular facet of Śaṅkara's voluminous writings has been the subject of a thorough series of studies recently by Plofker ([8,1] and other references cited there) and so we can leave it at that for the present, returning to specific points as the occasion arises.

Just as Nīlakaṇṭha returned in his old age, tangentially as it were, to some of the material he had set out earlier in his *Tantrasaṃgraha* by reexamining *Āryabhaṭīya* from the new post-Mādhava perspective – a conclusion supported by many citations and acknowledgements, e.g., "proofs [given by] mathematician-teachers such as Mādhava" (*mādhavādigaṇitajñācāryayukti*), at the end of the commentary on *Gaṇitapāda* –, so did Śaṅkara revisit *Yuktidīpikā*. Late in life he wrote – it is difficult to know precisely when but the work was left unfinished at his death – an extensive commentary ostensibly on Bhāskara II's *Līlāvatī* (mid-12th century), named *Kriyākramakarī*. Nīlakaṇṭha used *Āryabhaṭīyabhāṣya* partly to contextualise the Nila work within the Āryabhaṭan framework and, conversely, Mādhava to finally validate the cryptically expressed vision of Āryabhaṭa. Perhaps the choice of *Līlāvatī* as the object of his return to roots says something about Śaṅkara's own view of what Mādhava really achieved. *Līlāvatī* became the most popular of Indian mathematical texts, even more than the *Gaṇita* chapter of *Āryabhaṭīya*, but it is not on the same wavelength as the Nila school: the only overlap in matters relating to infinitesimal methods concerns the formulae for the surface area and volume of the sphere. Nevertheless *Kriyākramakarī* has long passages on the Nila results, including the power series ([20] has extensive extracts), but without establishing any organic links they may have with the appropriate parts of *Līlāvatī*; in fact a high proportion of its material on circle geometry is lifted verbatim from

Yuktidīpikā and so it does not need to be looked at independently of the latter.

All in all, one carries away the impression that, unlike Nīlakaṇṭha who used the new methods of Mādhava to illuminate the enigmatic *sūtra*s of Āryabhaṭa, Śaṅkara's gaze was turned the other way, seeking the seeds of the Nila revolution in the conventional wisdom of Bhāskara II. As we shall see, they cannot be found there.

Since most of the rest of this article is going to be about the contents of *Yuktibhāṣā*, only some general prefatory remarks on what makes it unique and worthy of careful study are offered here. The external attributes of this uniqueness are now well-known: it is in Malayalam, not Sanskrit – as the word *bhāṣā* in its name proclaims – and it is in prose. It is also now accepted that it is the prime exhibit for the case that Indian mathematics did not function on faith and authority (and some computation) alone but demanded adherence to a set of principles of validation[20] the application of which led to *yukti* (or *upapatti* in a slightly narrower sense) which I shall often simply call a 'proof'. That accounts for the first half of the book's name; indeed, in the entire book there are only two nontrivial (as it happens, highly nontrivial for its time) results (one of them actually a collection of results) for which the proofs are not given.

Less immediately obvious are the quality and style of presentation. The language is informal and down-to-earth, almost colloquial, the tone persuasive rather than professorial. There are no patronising mannerisms as with other authors; no proof is withheld as a challenge to the adept as often done by Bhāskara II or because it is either beyond their comprehension (Śaṅkara in connection with a combinatorial formula arising in the derivation of the sine series) or "self-evident to the intelligent" (Nīlakaṇṭha about the theorem of the diagonal). As for the substance the language conveys, it is developed in a manner very much to the point, sharp and logically well-structured. It is also markedly 'theoretical', emphasising principles and with very few illustrative examples provided to mitigate the rigour. There are even (rare) occasions when Jyeṣṭhadeva actually excuses himself for falling back on time-honoured conventional numbers as proxies – *parārddham* (10^{17}) for an unboundedly large number into which the unit tangent is divided (the π series) and 24 for an arbitrary number into which a quadrant of the circle is divided (the sine series). Overall there is a degree of mathematical insight, sophistication and taste that is quite surprising at first meeting – and not only in comparison with other contemporaneous texts – and of which we will meet instances as we go along. Especially noteworthy is the care devoted to the presentation of ideas which, in retrospect, we can identify as being truly original and profound and, conversely, the summary dismissal of methods which have no more than computational value.

As in many other Indian treatises, the opening chapter is devoted to the decimal place-value construction of positive integers and the rules of arithmetical operations with them.[21] Of particular interest to the present article is the list of names of the powers of 10, ending with *parārddham* (10^{17}) which will later be used as a proxy for a very large number, to be allowed to grow unboundedly in denominators as the means of introducing 'infinitesimals'.

Of chapters 2-5, the very brief chapter 4 on the 'rule of three' (*trairāśikam*) in its arithmetical context is of interest for its application later to the geometry of similar triangles. The proportionality of sides of similar triangles and the theorem of the diagonal (Pythagoras' theorem) are of course the two main pillars on which Indian geometry was built from the time of the *Śulbasūtra*, through Āryabhaṭa's trigonometry, right down to their adaptation by the Nila school to an infinitesimal setting. The Pythagorean theorem itself is dealt with in the opening section of chapter 6.

Chapters 6 and 7 are dominated by the mathematics of the power series, treated in great detail; the calculus computation of the area and volume of the sphere is tacked on at the end of chapter 7, after a digression on the properties of cyclic quadrilaterals (geometry *à la* Brahmagupta, nothing infinitesimal here). These two chapters will obviously be the main object of our study. Apart from the many novelties, what they convey powerfully is the care and attention paid to matters that we will now call analytic (as distinct from the numerically approximative): limits are carefully defined and implemented, quantities which are of a higher order of smallness are isolated and neglected and their contribution then actually shown to vanish in the limit; such issues were never broached in the pre-Nila texts because they never arose. When it comes to the meaning to be attached to infinite series, there are clear statements to the effect that they are exact if and only if they are not terminated; even the need to ensure that variables (tangent or cotangent) in which the expansion is made in different domains of the angle remain 'small' is addressed.

The calculus of the π series

All of chapter 6 of *Yuktibhāṣā* is devoted to the problem of determining the ratio of the circumference and the diameter of a circle exactly or to arbitrary precision. The aim is achieved in the development of the basic π series, which is therefore the centrepiece of the chapter, with every 'lemma' and 'proposition' needed in establishing the 'main theorem' taken up and proved in logical sequence. After its easy generalisation to the arctangent series, the chapter concludes with an exhibition of conceptual and technical virtuosity by asking for methods of overcoming the slow rate of convergence of the basic π series and finding them: through estimates of the remainder after truncation at a finite but arbitrary number of terms and by developing accelerated π series, the basic series modified by reordering the terms for faster convergence.[22]

Before taking up the π series, *Yuktibhāṣā* describes the classical method of calculating π to any given precision by approximating the circumference of a circle by the perimeter of a sequence of circumscribing regular polygons starting with a square.[23] The description ends with the passage[24] ([YB 6.2]):

> If, by the method for the production of the side of the 16-gon [from the 8-gon], [we] double and redouble the number of sides to the 32-gon and so on and if the number of vertices is increased beyond count (*asaṃkhya*), [it becomes] of the nature of a circle (*vṛttaprāyam*). Imagine this to be the circle. The diameter of this circle is the side of the initial square.

The passage is noteworthy for being the first explicit statement of the method of computing arbitrarily accurately the length of a curve (in this case the circumference) by a process of dividing it by a number n, approximating each of the resulting arc segments by a line (in this case the tangent through the midpoint of the arc) segment, adding them up and finally taking n to infinity (*asaṃkhya*). This is a sort of almost-calculus – it requires, in an essential way, the notion of a limit to be brought in, unlike in Archimedes' method of exhaustion. But *Yuktibhāṣā* does not pursue the method to the limit for the reason that it entails computing an increasing proliferation of square roots. What is infinitely (no pun!) more significant is the laying to rest, by means of the one word *asaṃkhya*, of a ghost that had haunted geometry for close to a thousand years: the belief that the 96th part of the circumference is equal to its sine (Bhāskaras' fallacy, see section 7 below).

As the standard against which to place *Yuktibhāṣā*'s method of obtaining the π series and as a natural introduction to the basic notions of calculus in practice, it is useful to set down, without being too pedantic, how the 'Gregory-Leibniz' series is treated in today's classrooms. For an angle θ lying between 0 and $\pi/4$, define $t := \tan\theta$, $0 \leq t \leq 1$. Then $dt/d\theta = 1 + t^2$. Invert both sides of this equation and use the formula for the sum of an infinite geometric series (or use the binomial theorem):

$$\frac{d\theta}{dt} = \frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots.$$

Now integrate term by term using the formula

$$\int_0^1 t^k dt = \frac{1}{k+1}$$

for $k = 2, 4, \cdots$ and we get the series.[25]

The steps in this proof are all familiar though the metaphysics behind them is not often explicitly noted in our textbooks:

i) Differentiation is (and, historically, always has been) a geometric process in its essence, that of finding the tangent at any point θ to the graph (local linearisation) of the function $\tan(\theta)$. In practice $dt/d\theta$ or $d\theta/dt$ can be determined in several apparently different ways but they all boil down to the geometry of the circle.

ii) Inverting the derivative makes t rather than θ the independent variable. The step takes advantage of the contingency that the derivative is an explicitly expressed function of t so that the integration is reduced to a quadrature.

iii) But the (indefinite) quadrature does not lead to a 'simple' function of t. It cannot: if it resulted in a rational function for example, then π would be a rational number. The recourse to the infinite series expansion works because we have prior knowledge of the values of integrals of (positive integral) powers 'from first principles'; in a sense the expansion reconciles the wish for easy integration with the irrationality of π.

iv) Integration, in general, is essentially an arithmetical process, that of adding up the slopes of the tangents at all points of the graph of the function.

v) Underlying everything is the fundamental theorem of calculus which is no more than the 'infinitesimalisation' of the common sense principle: the whole is the sum of its parts. As one would expect of such a general principle, it is of universal applicability and has many variants and generalisations. The simplest form, the one which is used here, expresses the mutually inverse relationship between the operations of differentiation and integration:

$$\int^y \frac{df(x)}{dx} dx = \frac{d}{dy} \int^y f(x) dx = f(y)$$

for any fixed lower limit of integration where f is a (real) function (satisfying some mild conditions to ensure that all the operations are well defined) of one real variable x.

vi) And of course we have taken advantage of the Cartesian equivalence of functions and their graphs to pass freely between the geometric and the analytic languages.

In their geometrical guise, the π and the arctangent series are examples of what came to be known as rectification – finding the length of a curve between two given points – in fact the simplest nontrivial example, the curve being an arc of a circle.[26] *Yuktibhāṣā* chooses to rectify one octant, perhaps through a feeling for the symmetry of the circle.[27] As shown in the figure, O is the centre of the unit circle, OA a radius, AB the unit tangent (half the side of the circumscribing square, tangent at A) and P a point on AB. Then length(AP) is $\tan\theta =: t$, with $\theta = \text{angle}(OA, OP)$. What is required is to find the length of the arc corresponding to AB (one-eighth of the circumference) as a number or, more generally, the arc corresponding to AP $(= \theta)$ as a function f of t. Thus t is the independent variable and the function f is arctan with $\theta = f(t)$ (see Fig.1).

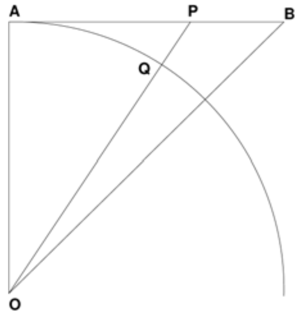

Fig. 1: Determining the ratio of the circumference to the diameter of a circle.

Yuktibhāṣā begins by dividing AB into many equal segments by marking points P_i, of length $1/n$ ($P_0 = A$ and $P_n = B$) where n is a large number: "[On the unit tangent], mark a certain number of points, very close, so as to divide it into equal intervals. As large as the number is, so accurate (*sūkṣmam*) will be the circumference" (YB 6.3.1). If the line OP_i intersects the arc at Q_i then, corresponding to the tangent segment $P_{i-1}P_i$ of length $1/n$, we have the arc segment $Q_{i-1}Q_i$. In modern notation we will write $\delta t = 1/n$ and $\delta\theta = \delta f(t) = f(i/n) - f((i-1)/n) = \text{arc}(Q_{i-1}Q_i)$. The key geometrical step now is to replace $\text{arc}(Q_{i-1}Q_i)$ by $(1/2)\ \text{chord}(Q_{i-1}Q_{i+1})$ = length of the perpendicular from Q_{i-1} to $OP_i = \sin\delta\theta$ (see Figure 6.5 in [13]).[28] In the limit $n \to \infty$ ($\delta t \to 0$), the half-chord tends to the tangent (of vanishing length) to the circle at Q_i, but the notion of tangency is not something that *Yuktibhāṣā* is concerned with.[29] The pertinent point is that, *as the half-chord gets smaller and smaller, its length approaches the arc length:* $\sin\delta\theta \to \delta\theta$ *as* $\delta\theta \to 0$ or, more faithfully to the text, $\sin(\theta/n) \to \theta/n$ as $n \to \infty$. Since $\delta\theta \to 0$

as $\delta t \to 0$, we can summarise how the geometry simplifies as n becomes very very large, as *Yuktibhāṣā* does in the last sentence of section 6.3.1: "If the segments of the side of the (circumscribing) square are very very small, these half-chords will be almost the same as the arc segments".

If we are asked to mark the birth of calculus by identifying one paradigm-changing insight, we cannot do better than cite this *principle of asymptotic linearisation.* The name will evoke the principle that underlies all of the Nila work on calculus but will also serve to distinguish it from the way infinitesimal quantities were introduced in European calculus. Blindingly self-evident as it may appear, this property of arcs and chords of circles eluded everyone who came between Āryabhaṭa and Mādhava as we shall see in section 7. Āryabhaṭa himself almost surely understood that the equality of the arc and the chord was only an approximate one so long as they both remained of nonzero length. That is the natural way to interpret the organisation and content of the verses of *Āryabhaṭīya* that prepare the ground for the sine table (*Gaṇita* 10, 11, 12), especially the superenigmatic verse 11. *Yuktibhāṣā* restates the principle in slightly different forms from time to time, but without any obvious insistence – maybe it did become self-evident once Mādhava had grasped it and put it to such splendid use.

The computation of the half-chord $\sin\delta\theta_i$ of the arc $Q_{i-1}Q_i$, before taking the limit of large n, is an example of Indian geometrical reasoning at its most typical, using some astutely chosen pairs of similar (right) triangles (see [13], figure 6.5). The proportionality of their sides (*trairāśikam*) then allows a simple calculation of the half-chords in terms of the 'diagonals' (*karṇṇam*) OA_i:

$$\sin\delta\theta_i = \frac{\delta t}{d_{i-1}d_i} = \frac{1}{nd_{i-1}d_i}$$

where $d_i :=$ length(OA_i). As the value of $\sin\delta\theta_i$ this expression is exact, valid for any value of n. The unfamiliar diagonals may be a distraction but they are easily got rid of by means of the theorem of the diagonal: $d_i^2 = 1+i^2/n^2$ (1 being the value of the radius), In any case, this geometrical result expressing $\sin\delta\theta_i$, the (linearsied) variation of the function θ of t at the ith point in terms of the variation of t, subsumes the 'differential' part of the problem.

Yuktibhāṣā actually carries out some steps of asymptotic approximation (by which will be meant from now on an approximation which tends to exactness with increasing n) before eliminating the diagonals, first by writing

$$\frac{1}{d_{i-1}d_i} = \frac{1}{2d_{i-1}^2} + \frac{1}{2d_i^2}.$$

This is one of the very rare steps for which the justification is not given but we can easily supply one. The differences between the two sides is $(d_i - d_{i-1})^2/2(d_i d_{i-1})^2$; since $d_i - d_{i-1}$ is of order $1/n$, the error is of second order of smallness and so is negligible when n such terms are added as will be done during the 'integration' part of the problem. To the same accuracy, the denominators d_{i-1}^2 and d_i^2 can be equated (this step is properly justified during the integration phase of the work, see below). The final step is to replace $\sin\delta\theta_i$ by $\delta\theta_i$, leading to the asymptotically exact differential relationship

$$\delta\theta_i = \frac{\delta t}{d_i^2} = \frac{1}{n(1+i^2/n^2)}$$

or, dispensing with the now superfluous i,

$$\delta\theta = \frac{\delta t}{(1+t^2)}.$$

What the geometry has accomplished in the limit $n \to \infty$ is thus the determination of the differential of θ as a function of $\tan\theta$.

It is a distinctive feature of the Nila approach to calculus that taking the limit $n \to \infty$ is delayed as late as is practical. *Yuktibhāṣā* has no way to find the exact sum of $\delta\theta_i$, namely $\sum_{i=1}^{n}(1+i^2/n^2)^{-1}$. So the first step it takes in integrating $\delta\theta$ is to expand the denominator into an infinite series, not by appealing to the binomial theorem but by means of a versatile technique of great antiquity, that of recursive refining (*saṃskāram* as it is called in *Yuktibhāṣā*):

$$\delta\theta_i = \frac{1}{n} - \frac{i^2}{n^3} + \frac{i^4}{n^5} - \cdots$$

with the idea of doing the summation over i term by term (as long as n is kept finite, this requires no justification).[30] But here also there is an obstacle; while the formulae for $\sum_{i=1}^{n} i^k$ were familiar already to Āryabhaṭa for $k = 1, 2, 3$, they were unknown territory for higher values of k (for general k they involve the Bernoulli numbers). And, once again, *Yuktibhāṣā* gets around the obstacle by identifying and evaluating the asymptotically dominant form of the sum.

The evaluation of these sums and the passage to the limit $n \to \infty$ are among the topics most carefully treated in *Yuktibhāṣā* (sections 6.4 and 6.5). Several deep ideas and methods pointing to future developments make their first appearance in these sections ([7]). Since good, faithful accounts in English and in modern notation and terminology of this material ([3,13]) are available, only a brief summary highlighting the calculus-related points is offered here.

Denote the exact sum of the half-chords by (all sums over i are from 1 to n)

$$S_n := \sum \sin \delta\theta_i = \frac{1}{n} \sum \frac{1}{d_{i-1} d_i}.$$

We have seen above that S_n can be approximated by

$$S'_n := \frac{1}{2n} \sum \left(\frac{1}{d_{i-1}^2} + \frac{1}{d_i^2}\right)$$

and further by

$$S''_n = \frac{1}{n} \sum \frac{1}{d_i^2}.$$

The second approximation is also asymptotically exact because $S'_n - S''_n = (1/2n)(1/d_0^2 - 1/d_n^2) = 1/4n$ since $d_0^2 = 1$ and $d_n^2 = 2$ for the unit circle (*Yuktibhāṣā*: "As the segment of the side (AB) becomes smaller, the one-fourth part ($1/4n$) can be discarded"). It is this final form of the approximation that leads, as we saw before, to the standard expression for the differential of θ with respect to t and hence to the discrete integration (*saṃkalitam*) formula

$$S''_n = \sum_{k=0}^{\infty} \frac{1}{n^{2k+1}} \sum i^{2k}.$$

The limit as $n \to \infty$ of the right side is therefore $\pi/4$.

Lacking an exact formula for the finite sum $\sum_i i^k$ for general k, *Yuktibhāṣā* resorts to evaluating the asymptotically dominant form of the known exact expressions for $k = 0, 1$, namely n and $n^2/2$. It then relates, by an elementary but clever rearrangement of terms, the dominant form for the case $k = 2$ to that for $k = 1$ and then, again, $k = 3$ to $k = 2$ and explains carefully how the procedure can be carried through for all powers of k. The end result is the *saṃkalitam* quoted earlier,

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1},$$

thus completing the evaluation of the integrals of positive powers.

The reduction of the sum of the kth powers to the sum of the $(k-1)$th powers is the first known instance of a proof by mathematical induction in India. Both the elaborate, step-by-step, description of the procedure and the accompanying explanations (YB 6.4) make it unmistakably clear that Jyeṣṭhadeva is aware that he is presenting a method of proof far removed from the traditional geometry-based techniques of summing series. (Many of the Nila mathematicians, including Jyeṣṭhadeva himself, were partial to these geometric techniques).

Of far geater interest in the context of calculus, however, is the rearragment trick mentioned above. As discussed in some detail in [7], it is a special case of the elementary identity

$$\sum_{i=1}^{n-1} (n-i)a_i = \sum_{i=1}^{n-1} \sum_{j=1}^{i} a_j$$

and is none other than a discrete version of the rule for integration by parts (the Abel resummation formula) used, if one wishes to make a point of it, in conjunction with the discrete fundamental theorem.[31]

After this resumé, it is in order now to try and benchmark *Yuktibhāṣā*'s approach to the π series against the points i) to vi), noted at the beginning of this section, that characterise its modern derivation.

i) Local linearisation is a geometric operation (even when expressed as the differentiation of an analytically given function), but there is no unique way of doing the necessary geometry. *Yuktibhāṣā*'s approach to linearising θ as a function of $\tan\theta$ may be original and superficially unfamiliar, but the essential geometric input is the orthogonality of the radius and tangent at any point on the circle.

ii) The geometry leads directly to $\delta\sin\theta_i$ as a function of i/n and then, in the limit $n \to \infty$, to $\delta\theta/\delta t$ as a function of t. The *saṃkalitam* $\sum_i \delta\theta_i$ is the discrete counterpart of quadrature.

iii) The expansion into an infinite series is identical even if the particular way of justifying it may have come from different sources. The suspected irrationality of π may have had a motivational role in the Nila school's turning to infinite series to pin down its 'exact' value ([6]). Their acceptance as being mathematically legitimate almost certainly had to do with the long familiarity with decimal numbers. In Europe, especially in the early work of Newton, the approach to infinite series was more 'functional', but Newton too modelled algebraic operations with infinite series on arithmetical operations on the decimal representation of integers (the tract *Die Methodis Serierum ...* in [23]).

iv) The idea of integration as the adding up of linearised local variations (differentials) of a function is essentially the same in India and in Europe. There are 'ideological' differences – such as the Indian resort to division by a large number instead of defining infinitesimals directly as in Europe – but they are of little mathematical consequence.

And then there are the two methodological novelties, not only in the historical sense but also as compared with the modern treatments: today's textbooks do not evaluate $\int x^k dx$ by induction on k; nor do they make use of integration by parts (though Leibniz did).

v) Since the limit $n \to \infty$ is taken at the end of all computations, the fundamental theorem of calculus has no obvious relevance. Its discrete counterpart is a self-evident triviality but there is a price to pay: in every calculation, all neglected quantities must be explicitly demonstrated to vanish when added up in the limit (in relation to what is retained). *Yuktibhāṣā* does this work with care. Once that is done, in each calculation, *saṃkalitam* is the Riemann sum/integral.

The postponement of the limit to the very last and the consequent devaluation of the fundamental theorem lies at the root of the one major difference in the metaphysical grounding of Indian and European calculus. This is an issue which will concern us later on.

vi) There was of course no Cartesian equivalence in Indian mathematics: everything that led up to calculus revolved around the notions of numbers and geometry, especially of the circle.

I conclude this section by noting a curious omission: *Yuktibhāṣā* says nothing about the possibility of π being an irrational number. In his *bhāṣya* of *Āryabhaṭīya*, Nīlakaṇṭha famously interprets Āryabhaṭa's qualification of his (rational) value for π as proximate (*āsanna*) as signifying that an exact value for it cannot be found since there is no common measure for the circumference and the diameter that will not leave a remainder in at least one of them. After the elaborate care with which *Yuktibhāṣā* addresses the need to include *all* the terms in the series for exactness (the last sentence of YB 6.3.3 and the last paragraph of YB 6.4.1 for example), one would expect a remark to the effect that π may not have a finite numerical expression, especially in view of the thinking of his own teacher. The reason for the omission may just be that there was no *yukti* available for Nīlakaṇṭha's conjecture (naturally!). Or, perhaps more probably, Nīlakaṇṭha had not yet thought through and written *Āryabhaṭīyabhāṣya* at the time *Yuktibhāṣā* was completed.[32]

The sine series: setting up and solving a differential equation

The π series is an enabling result. Without an accurate value for π, neither accurate sines as a table for discrete angles nor the sine as an exact function of the angle can be determined; that is the reason why the verse in *Āryabhaṭīya* giving its value as 3.1416 (*Gaṇita* 10) immediately precedes the two verses describing the preparation of the sine table. The order of presentation in *Yuktibhāṣā* (and in other texts) respects the logical order. So, having set down the highly accurate estimates for the truncation errors in the π series at the end of chapter 6 – no calculus is involved here – *Yuktibhāṣā* begins chapter 7 by recalling the values of the sine and the cosine for some standard angles as well as some of the basic trigonometric identities and symmetries. The serious business, that of coming to grips with an accurate/exact evaluation, begins after this.

There are three clearly demarcated themes in *Yuktibhāṣā*'s approach to the whole complex of ideas culminating in the power series for the sine and the cosine: the working out of the 24 tabulated sines of Āryabhaṭa at angles $m\epsilon, \epsilon := \pi/48, m = 1, 2, \cdots, 24$; Mādhava's second order interpolation formula for $\sin / \cos(m\epsilon + \delta), 0 < \delta < \epsilon$; and a very full account of the power series. They get contrasting treatments in the book. The tabulated sines are dealt with adequately, starting with Āryabhaṭa's own initial value $\sin\epsilon = \sin 225' = 225'$ and proceeding step by step, always subject to the approximation $\sin\epsilon = \epsilon$. But, apart from a perceptive passage cautioning against using the rule of three for arcs because of their curvature (YB 7.4.1), it does not seriously prepare the ground for the way the power series will be approached, except that sine differences are defined and utilised in the computation - the second differences do not make an appearance here. The computation is described in two pages of text (YB 7.4.2). All in all, one comes away with the impression that for *Yuktibhāṣā* the table itsef is worthy of attention mainly on account of its venerable past (*pūrvaśāstra*, YB 7.4.1).[33] There are, however, scattered remarks here and there in this part of the text (like the one on the effect of curvature) that connect to the infinitesimal thinking to come. Their significance is best appreciated after we are done, in this section and the next, with the 'technical' material.

The interpolation formula is taken up next, but so briefly and in such a dismissive fashion as to leave little doubt that Jyeṣṭhadeva considered it a distraction from his main purpose. Though this too has its part to play when it comes to assessing the calculus credentials of the material in chapter 7, it is a negative one as discussed in some detail in [7].

The complete derivation of the power series for the sine (and the cosine; I will often omit an explicit reference to the cosine in what follows) series occupies all of section 7.5 of *Yuktibhāṣā*, about 10 printed pages in Sarma's edition. In the differential part of the work, the scheme followed is exactly the same as in the case of the π series: divide an arc (the 'independent variable' θ) into a finite number of equal segments and determine geometrically the difference between the sines (the 'function') of neighbouring arc segments. But as we know now, the integral part of the problem does not lend itself to a solution by quadrature since the derivative of the sine is the cosine and a power series expansion of the cosine is equally unknown, being part of the problem. The way *Yuktibhāṣā* handles this difficulty signals an unexpectly original advance, even from our present perspective; it amounts (in the asymptotic limit) to solving the equations for the sine and the cosine simultaneously or, equivalently, showing that the sine satisfies the familiar second order differential equation, converting it into an integral equation (the fundamental theorem!) and solving the latter by an iterative method.[34] True to the Nila philosophy, what are actually solved are the corresponding difference equations in the form of multiple repeated sums, with frequent evocations of the asymptotic limit to be taken eventually.

It is evident from the language that Jyeṣṭhadeva had in mind, while working out the procedure for finding the differential (YB 7.5.1), a division of the first quadrant into the canonical 24 parts. But any reservation one might have on this account is dispelled immediately (YB 7.5.2) by an injunction to think of the arc segments as being "as small as one wants" followed by: "One has to explain in one (definite) way; that is why [I] have said [up to now] that a quadrant has twentfour chords"; the geometry of sine differences works for any arc cut into any number of equal segments. Accordingly, take an arc of the circle of unit radius subtending an angle θ at the centre and cut it into $2n$ equal parts. (*Yuktibhāṣā* makes an n-fold division, but also uses the midpoints of the segments in the geometry; dividing by $2n$ from the outset simplifies the description). The values of the functions sin and cos at the ith point of the division are thus $s_i := \sin i\delta\theta$ and $c_i := \cos i\delta\theta$ with $\delta\theta := \theta/2n$, $i = 1, 2, \cdots, 2n$. The usual clever choice of similar right triangles (see [3] or [13] for the details) then leads to the difference formulae

$$\delta s_i := s_{i+1} - s_{i-1} = 2s_1 c_i,$$

$$\delta c_i := c_{i+1} - c_{i-1} = -2s_1 s_i$$

for $i = 1, 2, \cdots, 2n-1$ with $s_0 = 0$. (The right side depends explicitly on n through $s_1 = \sin(\theta/2n)$).

These exact relations,[35] true for any $n > 1$, capture the essential geometry that goes into finding the derivatives of sin and cos, just as the exact relation $\sin\delta\theta_i = 1/(nd_{i-1}d_i)$ captures the geometry of $d\theta/d\tan\theta$ (section 4). With this, the geometry is done. As $\delta\theta$ is made "as small as we want", i.e., as $n \to \infty$, δs_i becomes $\sin(\theta+\delta\theta) - \sin(\theta-\delta\theta)$, c_i becomes $\cos\theta$ and $s_1 \to \delta\theta$ and we get the derivatives

$$\frac{d\sin\theta}{d\theta} = \cos\theta,$$

,

$$\frac{d\cos\theta}{d\theta} = -\sin\theta.$$

But, compared to the situation of the π series, the sine and cosine differences presented a formidable challenge: the right sides are not explicitly determined functions of i and hence the sum over i cannot be done directly even in principle; it does not reduce to a (discrete) quadrature. In the limit correspondingly, the formulae for the derivatives, viewed as differential equations for the pair of functions sin and cos, are (first order) linear homogeneous. Techniques for solving such differential equations came relatively late in European calculus (second quarter of the 18th century; Euler, d'Alambert) and were certainly unknown to Newton and Leibniz. *Yuktibhāṣā* solves these difference equations by a very original method (even by almost-contemporary standards). There are two parts to this method. The first is a formal rewriting that employs the fundamental theorem in its discrete version to turn the difference equations into discrete counterparts of integral equations. In the second part, the discrete integral equations are solved by an appeal to the method of *saṃskāram*, an infinitely recursive 'refining' of a judicious first guess leading, in the limit, to the two power series.

The details of this process are covered in a number of recent publications, in particular [13] and, in a form and notation adapted to the concerns of the present article, in [7]. So the very short outline given below deals only with calculus-related issues.

The actual execution of the procedure begins with a step that will be familiar to the modern reader, that of converting the two coupled first order difference (differential) equations into one second order equation. Accordingly, define the second order differences

$$\delta^2 f_i := \delta f_{i+1} - \delta f_{i-1}$$

for $f_i := s_i$ or c_i. Substitution from one of the first order equations into the other (no more geometry is required) leads to

$$\delta^2 f_i = -4s_1^2 f_i$$

which becomes, in the asymptotic limit, the differential equation

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

for $f = s$ or c. Thus what was required can be recognised by us as the solution of the (one-dimensional) differential equation for harmonic motion for appropriate initial conditions for sin or cos. *Yuktibhāṣā* finds just such a solution by, effectively, first solving the 'discrete harmonic equation' and then taking the limit.

The key conceptual input in *Yuktibhāṣā*'s solution is, not surprisingly, the discrete fundamental theorem. The text takes considerable trouble over this issue, devoting the second half of section 7.5.1 and most of 7.5.2 to explaining how the sine differences are obtained by adding up their second differences and the sines themselves by adding up their differences[36], even providing, untypically, an illustrative example by way of the resulting formula for the 8th sine when the quadrant is cut into 24 segments. (It is after this passage that there occurs the statement that the Āryabhaṭan 24 is invoked only in a manner of speaking, as a substitute for a generic number). The end result for an arbitrary point in a general division of the arc is

$$s_{2i} = is_2 - 4s_1^2 \sum_{j=1}^{i-1} (i-j)s_{2j}.$$

In particular, for $i = n$, the left side is $\sin\theta$ and the first term on the right is $n\sin(\theta/n)$. The second term, on the other hand, has exactly the same structure as the corresponding sum in the working out of the π series, the left side of the equation on p.22 above, with the coefficients a_i now identified with s_{2i}. We can therefore do a 'discrete integration by parts' as before to arrive at the result

$$s_{2n} = ns_2 - \sum_{i=1}^{n-1} 2s_1 \sum_{j=1}^{i} 2s_1 s_{2j},$$

valid for any n.

This completes the first part of *Yuktibhāṣā*'s method of solving the difference equation for the sine. What the apparently trivial reordering has achieved is to turn the original difference equation into a 'discrete integral equation' as is made absolutely clear to our modern sensibility once we take the asymptotic limit: $2s_1$ becomes the differential $d\theta$, $ns_2 \to \theta, s_{2j} \to \sin\phi$ for an angle ϕ lying bewteen 0 and θ and the equation becomes

$$\sin\theta = \theta - \int_0^\theta d\phi \int_0^\phi d\chi \sin\chi.$$

We could of course have got to the same integral representation by first taking the limit of the difference equation, i.e., by formally integrating twice (using the 'true' fundamental theorem of calculus) the differential equation. In the spirit of *Yuktibhāṣā*, it is appropriate to call both it and its finite version the *saṃkalitam* representation as the word *saṃkalitam* does duty for the finite sum as well as its limiting form.

As for the second part of the method of solution, the amazing fact[37] is that the finite *saṃkalitam* can be carried out for any n exactly by the repeated substition of the equation into itself. Thus, substituting for s_{2j} on the right of the *saṃkalitam* representation, we get

$$s_{2n} = ns_2 - 4s_1^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} js_2 + (4s_1^2)^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{l=1}^{j-1} \sum_{m=1}^{l} s_{2m}.$$

The process can be continued indefinitely, the general term in the expansion being $s_2(-4s_1^2)^k S_{2k}(n-k)$, where the coefficients are the famous sums of sums (*saṃkalitasaṃkalitam*), defined recursively by $S_k(0) = 0$, $S_0(i) = i$ and

$$S_k(i) := \sum_{j=1}^{i} S_{k-1}(j)$$

with values[38]

$$S_k(i) = \frac{i(i+1)\cdots(i+k)}{(k+1)!}.$$

The series terminates for any finite n after n terms since $S_k(0) = 0$.[39]

With this solution in hand, it is now straightforward to pass to the asymptotic limit. As n is made to tend to infinity, the number of terms in the solution also tends to infinity, $s_2 \to \theta/n$, $4s_1^2 \to \theta^2/n^2$ and the coefficient $S_{2k}(n-k) \to n^{2k+1}/(2k+1)!$. The kth term therefore approaches $(-)^k\theta^{2k+1}/(2k+1)!$ (the powers of n cancel out) and we have the sine series.

The solvability of the discrete harmonic equation does a disservice; it might appear at first sight that, once the solution is found, all that is involved is a straightforward passage to the asymptotic limit with no specifically calculus-linked step playing a role. This is an impression created by the delaying of the limit as late as possible and by the absence of an *ab initio* definition of infinitesimals, its purpose being served by the asymptotic limit $n \to \infty$. Following from these methodological preferences, a single limiting operation, imposed at the very end, suffices to take care of all infinitesimal aspects of the problem. The impression is strengthened by the 'accident' of the coefficient $S_k(i)$ being exactly computable. In these respects, the situation is different from that of the π series where simplifications valid asymptotically had to be made separately in the differential (in establishing $\delta\theta = \delta t/(1+t^2)$) and the integral (in summing powers of integers) parts. It is partly to counter this impression that I have from time to time interposed the infinitesimal versions of key steps in modern calculus notation in the account above. The conceptual affinity of each stage of the development of the series with the corresponding steps in European calculus as it evolved in its turn should thus have become clear. As yet another illustration of these parallels, let us note that the integral equation satisfied by sin in the limit is just as naturally amenable to solution by the *saṃskāram* technique, by the repeated substitution of the equation into itself, leading to the identity

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots + (-)^k \int_0^\theta d\phi_1 \cdots \int_0^{\phi_{2k-1}} d\phi_{2k} \sin\phi_{2k}$$

and hence, as k is taken to infinity, to the power series.[40]

In concluding this section, it has to be said that the order of my description of its contents, with its emphasis on first solving the difference equation, deviates slightly from that of the text. *Yuktibhāṣā* does follow the same path but carries out the sums of sums after replacing $\sin\delta\theta = \sin\theta/n$ by θ/n. It makes no difference computationally since the coefficients are still the same sums of sums $S_k(n)$, but the infinitesimal intent is thus made clear. Throughout section 7.5, there are frequent reminders of the limit eventually to be taken, not letting the reader forget that the final goal is not some fancy trigonometric identity for $\sin\theta$ in terms of $\sin\theta/n$, but an exact expression for $\sin\theta$ as a function of θ. As a sample of these declarations of intent we have, in connection with the reduction of the sum of second differences of sines to the sum of the cosine differences (YB 7.5.2, right after the disclaimer that $\pi/48$ is only a conventional number and should be replaced by an angle "as small as we want"), the passage

> Since the arc segment is small, the *śarakhaṇḍayogam* (effectively the sum of the cosine differences) at the midpoints of the [equal] arc segments is almost equal to the same sum at their ends. So we can suppose them to be the same. The smaller the segment, the more accurate (*sūkṣmam*) the [resulting] sine. Taking the arc segment to be one part in *parārddham* (10^{17}) of one minute, multiply by the same number (10^{17}), carry out the sum and then divide by the denominator (again 10^{17}). It will be almost equal to the sum carried out without multiplying by the denominator.

'Denominator' here plays the same role as n (multiplied by the number of minutes in the arc) in our notation. As elsewhere in the text, it is clear that *parārddham* is code for a very large number. Sections 7.5.3 (the values of the coefficients $S_k(n)$) and 7.5.4 (successive refinements of $\sin\theta$) have several other similar comments. Particulary striking is a passage which comes in the middle of section 7.5.3:

> If the arc segment is taken as extremely (or endlessly, *atyan-*

> *tam*) atomic, the chord (sine) will become accurate. Thus if units (*rūpam*) which are of the nature of zero (*śūnyaprāyam*)[41] are added one at a time to the terms, the number (the sum) hardly changes.

Evidently, the treatment of the sine series is less easily slotted into our standard methods of doing calculus and the steps involved follow a less linear trajectory than in the π series. There are after all more ways than one of solving a differential equation and the one followed by *Yuktibhāṣā*, though natural from its 'recursive' perspective, is not what we are used to think of as the most elementary of them.

Very elementary calculus: the sphere

It is once again useful, as a point of reference, to start with the simple derivation of the surface area A of a sphere of radius R as given in a school textbook. Choose a great circle (the equator) and consider a strip of infinitesimal width $d\theta$ between circles of latitudes θ and $\theta + d\theta$ in one hemisphere. Its area is the product of the arc segment ($Rd\theta$) and the circumference of the latitude circle through θ ($2\pi R \sin\theta$):

$$dA = 2\pi R \sin\theta R d\theta.$$

Integrating over the hemisphere, we get

$$\frac{1}{2}A = 2\pi R^2 \int_0^{\pi/2} \sin\theta d\theta = 2\pi R^2.$$

Implicit in this most elementary application of calculus to geometry in dimension higher than 1 is that the student understands that the strip is not exactly a rectangle for a finite width $\delta\theta$, but tends to one as $\delta\theta \to 0$, i.e., that the deviation of the area of the curved strip from that of the (linearised) rectangle is of the second order of smallness. As befits the first ever account of this particular way of finding the area of the sphere, *Yuktibhāṣā*'s elucidation of the infinitesimal geometry that goes into it is carefully detailed. Where it differs from the school proof is in the integration: effectively, what it chooses to do is to integrate the sine by integrating its second differential, i.e., by partially solving the differential equation. Once again, the choice has something to tell us about its attachment to tried and tested methods and is announced at the very beginning of the section (YB 7.18) dealing with the problem:

> Now [I] describe [how], combining the two principles (*nyāyam*) [already] explained, [namely, that] from the sum of the sines (*piṇḍjyāyogam*) arises the sum of second differences of sines (*khaṇḍ-āntarayogam*) and [that], knowing the diameters at one place, we can apply the rule of three to any [other] chosen place (or, in a slightly different reading, 'as we please'), the area of the surface of a sphere will be produced.

The two principles are the proportionality of the second difference of the sine to the sine itself and the proportionality of the diameter of the slice through a given latitude ("place") to the sine of the latitude. The former of course is the breakthrough that led to the sine series.

In addition to Sarma's ([4]), a lightly annotated and more literal English translation of the whole of section 7.18 is available in [6]. It is sufficient here therefore just to indicate the main points so as to let the parallels with the standard treatment show themselves. The first thing to strike the reader is how graphic the detailed instructions for setting up the geometry – the equator and the latitudes as well as a great circle orthogonal to these, i.e., a meridian whose arcs measure the angle θ of the latitude – are. Thus, for computing the area between two successive latitudes (obtained, as always, by an equal division of the meridian into segments of angle $\delta\theta$), we are asked to "imagine that the circle-shaped gap between two circles is cut at one place, removed and then spread out". From the resulting very narrow trapezium, we are then to "cut out the [triangular] part outside the altitude [through one vertex], turn it upside down and transfer it to the other side", resulting in a rectangle (Figure 7.21 of [13]).

The estimation of the area of the strips ("the gaps"), $\delta A = 2\pi R^2 \sin\theta\delta\theta$, and their summing up ($\delta\theta$ is $\pi/2n$ and θ is $i\pi/2n$ at the

ith latitude, as in all earlier examples) parallel what we do today and need no further comment. And, as earlier, the limit $n \to \infty$ is taken at the end. But, having gone through this key step painstakingly several times, the explanation is brief, not bothering for example with division by *parārddham* and its use as a substitute for an unbounded number. But throughout this short section, we are reminded from time to time that the gaps are "small", e.g., just before concluding: "Because of smallness [of the arc segment], the width [of the gap] is almost equal to the full chord."

It is when it comes to the actual evaluation of the sum/integral that *Yuktibhāṣā* adopts a slightly curious procedure. It amounts to writing

$$\int \sin\theta d\theta = -\int \frac{d^2 \sin\theta}{d\theta^2} d\theta = -\frac{d\sin\theta}{d\theta} = -\cos\theta$$

instead of using directly the integral of $\sin\theta$ or, rather, its discrete version:

$$-2\sin\frac{\pi}{2n}\sum_i \sin\frac{i\pi}{2n} = \sum_i \delta\cos\frac{i\pi}{2n} = \cos\frac{\pi}{2} - \cos 0$$

in the limit. What makes this curious is that it was already known (and used in the derivation of the second difference equations) that the first difference of the cosine is the sine. On a lighthearted note, Jyeṣṭhadeva's predilection for reducing the solution of a simple problem to a more difficult one already solved perhaps only goes to confirm his credentials as a mathematician.

The volume of the sphere is obtained by following the same philosophy, of slicing through the latitudes, computing the volume elements (of the slices) and adding them up. The only (small) surprise is that the work begins by determining the area of the circle, known since a very long time, by a method, also probably going back to Āryabhaṭa, which can aptly be called 'visual infinitesimal geometry' (Figures 7.22 and 7.23 of [13]).

The surface and volume formulae have an interesting history which, as will be argued in the next section, can be used to throw some light on the evolution of infinitesimal ideas and the antecedents of calculus in India.

Sources

The 33 *sūtra*s of the second chapter (*Gaṇitapāda*) of *Āryabhaṭīya* constitute a compendium of the essential mathematics used in astronomical calculations. Of these, of special interest to us is a sequence of four consecutive verses from which flowed the mathematical ideas and, to an extent, the techniques that culminated in Mādhava's sine series. Over the intervening 900 years, the evolution of geometric/trigonometric methods followed an erratic path in more than one sense. A critical look at the high points of this trajectory is thus called for, both as a historical necessity and as a precondition for the evaluation of what Mādhava achieved.

The four verses are

Gaṇita 9 (second line): "The chord of one-sixth of the circumference is equal to the radius".

Gaṇita 10: "62,832 (the number is composed from standard number names, though in a convoluted way, presumably for metrical reasons) is the proximate (*āsanna*) circumference of a circle of diameter 20,000".

Gaṇita 11 is very ambiguously worded. A direct reading is:

> Cut (can also be read as 'cut equally') a quadrant of the circle. Half-chords of equal arcs, according to one's wish (*yatheṣṭam*), [are found] from triangles and quadrilaterals, by the radius.

A more fluent but still faithful reading will be:

> Divide equally, as one pleases, a quadrant of the circle. The half-chords (sines) of equal arcs so formed can be determined from the radius by means of triangles and quadrilaterals.

Gaṇita 12 is the verse that gives the formula for the differences of the 24 canonical sines in the first quadrant, $\sin m\pi/48 - \sin(m-1)\pi/48, m = 1, \cdots, 24$. The resulting numerical table is given mnemonically in verse 12 of the first chapter, *Gītikāpāda*.

Gaṇita 9 implies in particular that $\sin \pi/6 = 1/2$. It is very likely that Āryabhaṭa computed the first entry of the table (and possibly "some others") starting with this value and by means of the formula for the sine of the half-angle[42] and that this is the reason why he starts off with the regular hexagon.

The first thing to note about *Gaṇita* 10 is that a value for π accurate to five significant figures is necessary for the resulting values of sine differences in the table to be accurate to a minute of arc, which they are.[43] The method employed to get to this value remains unknown. A good guess is that it is based on approximating the circumference by the perimeter of the inscribed regular 96-gon, followed by some computational tweaking of the kind that was already known in Greece and in China, in other words the method outlined at the beginning of chapter 6 of *Yuktibhāṣā* (before it takes up the determination of π "without the use of square roots"), but starting with the inscribed hexagon and doubling the number of sides four times – the square cannot be used to get to the 96-gon. More relevant for us is the fact that more precise values for sines for any division of the circle cannot be obtained without a more accurate π. It is not a surprise then that the method of finding an infintely accurate value for π via the series comes before the treatment of the sines, in *Yuktibhāṣā* as well as in other texts of the Nila school.

The use of the word *āsanna* (which has a more sharply defined meaning than 'approximate' as it is often loosely translated; Sarasvati Amma ([3]) is careful to render it as 'proximate', close but not quite equal to) has been commented upon by at least two authoritative *bhāṣya*s, those of Bhāskara I (629 CE) and Nīlakaṇṭha. Bhāskara's explanation is vague and clumsy (for translations see [26] and [20]). In contrast Nīlakaṇṭha's, which of course came after the π series was discovered and its mathematical and epistemological implications had the time to sink in, is crystal clear. He says that the value of π can only be given as *āsanna* because it is an irrational number and sets down, in the process, an impeccable criterion for irrationality: that the circumference and the diameter are incommensurable (a translation of the relevant passage of *Āryabhaṭīyabhāṣya* will also be found in [3]).

The interesting question for us is: was Nīlakaṇṭha's conjecture made on the strength of the infinite number of terms in the π series? That seems unlikely since he himself, in the same work, illustrates the method of *saṃskāram* by applying it to infinite geometric series expansions of rational numbers. Alternatively, did the suspicion that π could not be written exactly as a fraction drive the search for its expression as an infinite series of fractions? We may, alas, never know the answer. What we do know is that every aspect of the work that goes into the final product that is the π series – the initial conceptual breakthrough of linearisation, the basic geometry and its infinitesimalisation, the reordering trick, the use of induction as a proof technique – was original, not to be met with even in some primitive form in the *pūrvaśāstra*, including *Āryabhaṭīya*. Indeed the whole approach via series and without taking square roots is a repudiation of Āryabhaṭa's own method of approximation. This cannot quite be said of the sine series, anchored firmly as it was in the methods of Āryabhaṭan trigonometry.

Gaṇita 12 does not say that 24 is the choice made for the number of arcs in a quadrant, something of a surprise after the phrase "as one pleases" of the preceding verse. It also seems that commentators differ among themselves on how exactly it is to be read literally. But as Shukla observes ([24]), with one exception (Nīlakaṇṭha), they all agree on its meaning which in our notation is summarised in the equation

$$\delta s_m = s_1 - \frac{1}{m}\sum_{i=1}^{m-1} s_i, \qquad m = 1, \cdots, 24.$$

The initial value s_1 is supposed known and, to an accuracy of $1'$, it is the value of the angle itself, $\sin \pi/24 = \pi/24 = 225'$. The formula as it is given depends on this approximation and also on $s_2 - s_1$ being 1 in these units and in this approximation ([25], [24]). As we shall soon see, these two coincidences caused much confusion in the minds of Āryabhaṭa's followers. Indeed, we have to wait for the post-Mādhava texts, *Yuktibhāṣā* (as already noted) and, very explicitly, Nīlakaṇṭha's *Āryabhaṭīyabhāṣya*, to see these confusions finally banished. As in the case of π, Nīlakaṇṭha's *bhāṣya* reads a little more into the verse than is actually there and gives a geometric procedure that is free from any approximation; though he sticks to $n = 24$, it is the same as that followed by *Yuktibhāṣā* in its approach to the sine series. Once again, it is as though he is using Mādhava's work to validate Āryabhaṭa's vision; the final formula $\delta^2 s_m = (s_m/s_1)\delta^2 s_1$ is equivalent to the discrete harmonic equation.

In summary, the most reasonable interpretation of the choices made in the compiling of the sine table (taken together with the numbers of *Gītikā* 12) is that $2\pi/96$ became the angular unit for reasons of astronomical necessity, that of getting the sines accurate to one minute of arc. The precision with which π was computed was similarly dictated.

If this was indeed the case, Āryabhaṭa's followers did not get the point. Bhāskara I's *bhāṣya* (629 CE) of *Gaṇita* 11, written only 130 years after *Āryabhaṭīya*, has an astonishing (no other word will do) passage (Plofker's translation [20]; see also [26]):

> It is proper to say that a unit arc can be equal to its chord; even someone ignorant of treatises knows this; that a unit arc can be equal to its chord has been criticized by precisely this [master].
>
> But we say: An arc equal to a chord exists. If an arc could not be equal to a chord then there would never be steadiness at all for an iron ball on level ground. Therefore, we infer that there is some spot by means of which that iron ball rests on level ground. *And that spot is the ninety-sixth part of the circumference.*[44]

It may seem hard to credit that this – that a circle is in reality a 96-sided regular polygon – could have been said by an astronomer in the long tradition of the Indian preoccupation with the geometry of the circle but, like the Aristotelian denial of instantaneous motion, it marks perhaps a natural stage in the progression from the finite to the infinitesimal. In any case, it could not have been the view of just one astronomer, however influential he may have been;[45] $2\pi/96$ became not just a practically convenient unit but the universally accepted 'quantum' or 'atom' of angle below which no one, it seems, dared to venture.

The misunderstanding had serious consequences, nowhere more so than in the reading of *Gaṇita* 11 – and it is a difficult enough task as it is. To start with, while the role of (right) triangles is obvious in all the trigonometry that followed, no commentator, traditional or modern, has found a convincing use for the quadrilaterals mentioned in the stanza in making the sine table; so one is in good company in not trying to make sense of it. For us, the key word in any attempt to read Āryabhaṭa's mind has to be *yatheṣṭam*, translated easily enough as "as one wishes" (or "arbitrarily" as it is used in today's mathematical discourse). Its sense is certainly not circumscribed by the number 96 and hence not exhausted by Bhāskara I's computation of the canonical sines by subdivision.

If we read *yatheṣṭam* in the sense "divide the quadrant by *as large a number* as one pleases" which is grammatically legitimate, the approximate equality $\sin 2\pi/n = 2\pi/n$ only gets better and the formula of *Gaṇita* 12 is still valid (with a minor change since the difference of the first two sines is no longer 1) as is the differential method of deriving it. It is in fact the only available method for computing a sine table starting with an arbitrary but sufficiently small angle. It is also elementary and efficient – no square roots. The only bit of knowledge not explicitly stated in *Āryabhaṭīya* that we have to attribute to the "master" in support of such an interpretation is that *he* knew no arc of non-zero length is equal to the chord but only approaches it as they both become smaller. Indeed Bhāskara I's repudiation of the "master" in the quote above supports such an attribution.

Gaṇita 10, 11 and 12 in this perspective thus mark the first articulation of a certain trigonometric vision, that of getting to an arbitrarily accurate sine table in arbitrarily fine steps with the help of an arbitrarily precise value for π. In Mādhava's mind that vision got transformed into an analytic one, that of determining the functional dependence of the sine on the angle. The numerous reminders in *Yuktibhāṣā* that the process of infinitesimalisation is to be continued indefinitely we may take as a tribute to this vision and as evidence of its fulfilment.

Five hundred years after Bhāskara I, we have another example of the staying power of the 96-fold division, this time in the work of his even more illustrious namesake, Bhāskara II. Bhāskara II has a method in two parts (in *Siddhāntaśiromaṇi* and his own explanatory notes on it) for the surface area of the sphere, resulting in the correct formula. The first part is identical with the method followed later by *Yuktibhāṣā* – slicing through latitudes, etc. – and surely served as its model, up to the point where the area of the region between two latitudes is computed by assuming it to be a trapezium, *except that the number of latitudes is fixed at* 24. The explanation stops abruptly at this point with the declaration that the area of the sphere is the product of the circumference and the diameter. The second part supplies the missing details (see [3] and [27] for these as well as translations). The end result is, not surprisingly, that the area is the sum of sines of the 24 canonical angles (with some adjustment for the end points). Bhāskara just sums them numerically using the table and verifies that it is very close to the announced formula (see [27] for the numbers).

At one point in the explanation, Bhāskara does say that more chords will result in more annular regions but there is no evidence that he actually resorted to a finer division; that would have required a finer sine table and a more accurate π. And there is of course not the faintest suggestion of making an infinitely fine division. How then did he arrive at the neat formula to which his numerical answer was only an approximation? In looking for a plausible answer, it is perhaps useful to remember that all kinds of correct but illegitimate results for circles and spheres can be derived just from looking at polygons and polyhedra if only Bhāskara I's fallacy – that $\sin \pi/n = \pi/n$ for some n – held. The perimeter of a regular n-gon for example would be $2nR\sin(2\pi/2n) = 2nR\pi/n = 2\pi R$, independent of n, and the area of a regular $2n$-gon would be, similarly, πR^2 where R is the radius of the circumscribing circle. A circle would really be indistinguishable from a polygon.

Let us conclude with two remarks of a historical nature on the technical means put to use in the development of the body of new knowledge that Mādhava created. The first is the geometry. As is well recognised now, the founding step of Āryabhaṭa's trigonometry was the association of an arc with its *jyārdha*, half the chord of twice the arc. More instructively, an arc subtending an angle θ at the centre is associated with the right triangle defined by radial lines through its two ends – one of them will cut the chord of twice the arc perpendicularly at its midpoint. The short sides of the right triangle so defined are then $\sin\theta$ and $\cos\theta$. For an Indian geometer, this was a natural thing to do. Right triangles and their Pythagorean characterisation (the theorem of the diagonal and its converse – as a geometric fact, not only as the existence of Pythagorean triples of integers and rational numbers) date back to the earliest of the *Śulbasūtra* (ca 8th century BCE), those of Baudhāyana and Āpastamba (verses 1.12 and 1.4 respectively). So does the other mainstay of Indian geometry, similar right triangles and the proportionality of their sides (Āpastamba 19.8) ([28]).

The second remark concerns the generation of arbitrarily small quantities by the use of unboundedly large positive integers as denominators. Decimal counting in India is of even greater documented antiquity than geometry. The earliest ever Indian text, *Ṛgveda* (ca. 1100 BCE or slightly earlier), though far from being a 'scientific' work, is abundantly rich in the names of numbers formed by the application of strict grammatical rules of nominal composition, attesting thereby to a mastery of decimal place-value enumeration ([11, 10]). The highest power of 10 occurring in *Ṛgveda* is 10^4 with the name *ayuta*. Within a very short time there appeared lists of names of powers of 10 going up to 10^{12} (then called *parārdha*) and 10^{19} (both in the *Taittirīya Saṃhita*) and, by the beginning of the common era, of such enormously high powers as to be of no remotely practical use – the concept of numerical infinity was already very much in the air. *Yuktibhāṣā*, following Bhāskara II's *Līlāvatī*, stops at *parārdha* (which by now denotes 10^{17}) but it lets us know that this was a matter of expediency. Right after the list comes the marvellously evocative sentence: "If [we] endow numbers with multiplication [by 10] and place-variation, there is no end to the names of numbers; hence [we] cannot know [all] the numbers themselves and their order" (YB 1.2).

The metaphysics of calculus: two cultures

Technical innovations of originality and power generally end up by becoming part of the conceptual fabric of the discipline. The idea of 'division by infinity' as a means of producing vanishingly small quantities is a case in point where calculus is concerned. For the Nila mathematicians, coming from a tradition comfortable with decimal numbers (to which "there is no end") since two millennia and longer, this was routine, a transposition of an elementary arithmetical operation to geometry. To fully appreciate the vital role of decimal numeracy in the genesis of the Nila calculus, it is enough to look at the struggle European calculus had with the *ab initio* notion of an infinitesimal. Newton's early writings (the first two volumes of the "Mathematical Papers" [29,30]) are ample proof of his preoccupation with the question before he settled on the idea of fluxions, finally giving them, in the *Principia*, a purely geometric and quasi-axiomatic formulation.[46]

When it comes to the geometry, the situation is the reverse. From its beginnings in the first cord-compass constructions of the *Śulbasūtra*, Indian geometry remained circle-bound. The well-documented early history of European calculus needs no rehashing here except to note that it too had its beginning in geometry. The geometric culture of 17th century Europe, descended directly from its Classical antecedents, was however far wider and deeper than its Indian counterpart of the 14th-15th centuries. The influence of that richness on the genesis of calculus, with the focus on properties of a much larger class of curves than circles – maxima and minima, tangents and normals, local curvature, rectification and quadrature, etc. – is already evident in the work of Newton's predecessors, most notably Fermat.

But of even greater impact was the Cartesian revolution which came (fortuitously?) at the same time, liberating geometry forever from the confines of physical space and allowing a precise and quantitative approach to the study of geometrical objects which challenge an immediate intuitive apprehension, for instance those in dimension higher than the physical 3 and, in the plane, curves of relatively high degree. Most significantly, it helped turn calculus into a set of algorithmic rules which did not require the genius of a Newton or a Mādhava to be put to productive use.[47] Calculus became the study of functions and Newton's fluxion became the derivative in its currently recognisable definition: $f'(x) := \lim_{y\to 0}(f(x+y) - f(x))/y$.

Given their common roots in geometry, we can make a beginning in understanding what mainly distinguishes the two cultures by saying that in India calculus was born from the impact of decimal arithmetical thinking on geometry while in Europe it was algebra – which itself began as an extension of arithmetical operations to "affected quantities" – that was the transformimg agent. For a graphic illustration of what this meant in practice, we have only to look at Newton's approach (1669, *De Analysi . .* [30]) to the sine series and contrast it with Mādhava's. He begins like Mādhava by differentiating the sine and would like to expand the derivative $(1 - \sin^2)^{1/2}$ binomially but does not know how to integrate the resulting powers of the sine with respect to the angle. So he inverts the derivative and expands $(1 - \sin^2)^{-1/2}$. Now he only has to integrate powers of sine with respect to itself which of course was no problem. But the result is an infinite series for the angle in powers of its sine and that is not what he was primarily after:

> If it is desired to find the sine from the arc given, of the equation $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 \cdots$ found above (z is the angle and x is its sine), I extract the root, which will be
>
> $$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 \cdots$$
>
> Let it be noted here, by the way, that when you know 5 or 6 terms of those roots you will for the most part be able to prolong them at will by observing analogies.

Newton actually solved algebraically ("extracted the roots") the equations resulting from truncation after the first few terms successively before resorting to "analogies" – "truly a physicist's proof".[48]

A linguistic parallel with the arithmetic/algebra dichotomy would be to say – since we are speaking of Indian thought – that numbers are defined by their semantic significance whereas the symbols of algebra are syntactically defined, taking on whatever meaning we choose to assign them, subject only to a set of "rules without meaning" (in Frits Staal's perceptive phrase [31]) applied mechanically. Divorcing structure from sense – as algebra does – bestows freedom, freedom to apply the rules that define the structure without concern for the sense of the objects to which the rules apply. And from that freedom comes the power of abstraction which in turn is the precondition for generalisation. The amazing growth of mathematics in all its great variety from Descartes' days to our own is an eloquent testimony to this truism. In the realm of calculus, the landmarks crossed by the ever widening waves of generalisation already in its classical phase will include, among many others, calculus in more than one (real) variable and its extension to functions of complex variables. Subsequently, what was considered geometry was itself transformed by calculus, bringing in the greatly more general spaces of modern differential geometry into its fold. Ever more sophisticated analytic methods were brought to bear on their investigation, with the fundamental theorem adapting itself naturally to every changing circumstance to become, as of today, Stokes' theorem in its modern connotation. D'Alembert's "métaphysique du calcul" has, in today's perspective, become the enormously more general statement that calculus is the study of mathematical objects which can be given a topological structure (so that 'vanishingly small', and more generally the notion of a limit, can be defined) and a linear structure (for the operation of addition without which derivatives and integrals cannot be defined).

Nothing remotely resembling the explosive growth of the discipline, even in its premodern phase, say the two hundred years between Newton and Weierstrass, happened on the banks of the Nila in the two hundred years separating Mādhava from the definitive decline of the school he founded. Various reasons, some of them political, can be put forward for this loss of collective intellectual vigour, but the dominant cause must be sought internally. Firstly, every single one of the Nila savants was first an astronomer, and a mathematician only by necessity.[49] But more than this 'applied' orientation, the general conservatism that marked Indian intellectual life after the century of enlightenment, the 6th, surely acted as a brake on originality of thought. Mādhava broke away from the beaten track – as Bhāskara II did not – and that is the true measure of his stature as one of the greats. But despite that brilliant legacy and despite also his own exhortations not to be a slave to received wisdom, what Nīlakaṇṭha turned to in his wise old age was the thousand year old work of Āryabhaṭa.

Minimal algebra also meant a lack of interest if not an aversion to abstraction and generalisation.[50] In not one of the main Nila texts is there an intimation of the universality of the concepts and methods they had in hand.[51] Apart from this general indifference, it is easy to cite specific instances of interesting potential generalisations being missed which to a 'structural' mindset would have been obvious. The first is of a certain historiographic interest since it has been an issue in the 'calculus or not?' debate. Once the sine series was found, its generalisation to the Taylor series would have required only two easy steps: a recognition that a circle has no preferred point ("a uniformly round object" as *Yuktibhāṣā* characterises a sphere) and the addition formula for the sine ([6]). What we have instead is the 2nd order interpolation formula of Mādhava which Śaṅkara extended to higher orders, resulting in a series which is wrong ([7]). A second instance has nothing to do with calculus and is actually a case of specialisation rather than generalisation but it exemplifies the same mindset. *Yuktibhāṣā* has a clear exposition of Brahmagupta's theorems on cyclic quadrilaterals, in particular one which expresses the area in terms of its four sides (YB 7.15.1-5). After that comes the corresponding formula for the triangle, obtained not by putting one side equal to 0 but by an independent (and long) geometric proof (YB 7.15.6).

But regret for missed opportunuties should not blind us to the true achievement of Mādhava. A careful reading of *Yuktibhāṣā*, with the eye and the mind open to its many illuminating side remarks, leaves little room for doubt that that achievement was the formulation of what we now consider the fundamental principles of calculus. Not only are they presented and explained with attention to detail and warnings against pitfalls, but many of the collateral steps in the quick progress of classical European calculus – and some which came relatively late – were also anticipated. As a reminder that the motivating force came from the need to be able to handle curves, here is a final quote from *Yuktibhāṣā* (section 7.4.1, after explaining that the use of the linear approximation for arcs will lead to gross error):

> The reason for this: the second arc [is] twofold the initial arc. ... The second chord is not twofold the initial chord, the third chord is not threefold, and so on. The reason for this: the initial arc has no curvature [and is] almost equal to the chord since the *śaram* $(1 - \cos)$ is small. So do not apply the rule of three (linear proportionality) to the arc because the result

will be gross (*sthūlam*).

A fair summing up then would be to say that, lacking the "metaphysical" breadth that characterised the European response to the same challenge, all these brilliant insights never broke free from the concrete and the particular. The fundamental theorem, for instance, remained a non-issue: it is a dispensable luxury so long as interest is confined to a handful of problems in which the infinitesimal limit can be handled case by case. The recognition in Europe of its foundational importance, already by the pioneers, is in itself a tribute to the universality of their vision.

It is appropriate to bring this article to a close by taking up briefly two issues of a historiographic nature mentioned in the introduction. First, are there any hints of an infinitesimal mode of thinking in the pre-Nila writings in mathematical astronomy? The case that there are indeed such indications has most explicitly been made in a recent paper ([27]), on the basis primarily of methods for finding corrections to planetary motions attributed to the 9th century astronomer Mujjāla/Mañjula and Bhāskara II himself. The credibility of the evidence depends on the interpretation of a particular piece of phraseology in their writings, evocatively rendered in [27] as "instantaneous velocity", a concept which has a long and contentious history in Europe going back at least to Aristotle. If this rendering is correct, these allusions will constitute an anticipation of the Newtonian view of dynamics as calculus, with position as a function of the primordial variable, time, and 'velocity' as the derivative. The texts however do not appear to support such a sharp reading.[52] In all of the instances cited, the variations of planetary parameters being discussed are over an interval of time, generally one day. The phrase "instantaneous velocity" is used as the translation of *tātkālika gati*; but *tatkāla* has the literal (and, in the context, natural) meaning of"that (designated) *kāla*" and *kāla* itself is most commonly used for an interval of time.[53] Besides, the passages make it clear that the corrections discussed refer to changes over a day, differences rather than differentials.

The same article ([27], section 17) also states that the notion of a time derivative is present in the writings of Nīlakaṇṭha and Acyuta. On the face of it this is more plausible since the infinitesimal had by then become an integral part of Nila thinking. It would also be exciting since, in addition to the implied acceptance of a non-geometric quantity like time into the fold of calculus, the functions involved are far from elementary. But, once again, the passages do not compel a reading going beyond a finitistic first order correction. The case for a thorough critical reading of the late Nila astronomical texts continues to remain a strong one.

The tendency to overinterpret ancient knowledge has always been a natural occupational hazard for the historian of science, saturated as his or her mind is with all the progress that has been made since – we have only to think of Nīlakaṇṭha's reading of *Āryabhaṭīya* as an honourable example. Paradoxically, viewing the scientific achievements of the past from today's vantage point can also induce the opposite error, that of undervaluing them by holding them up to the impossibly greater generality and depth of contemporary science.[54] To a certain extent, the distorting glass of progress may well account for the other point made in the introduction: why is there a debate about whether the mathematics that Mādhava created is calculus or not? Part of the answer must be: sheer incredulity, that the fine flower of classical calculus had already bloomed in a distant corner of the world some centuries earlier. Partly it is the reliance by historians on the wrong text, *Yuktidīpikā*, and the wrong material from that text, the interpolation formula to which it gives a great deal of importance[55]; the linguistic inaccessibility of *Yuktibhāṣā* until very recently only compounded the difficulty. But a more fundamental reason surely is mystification at the absence of what we consider the pillars of classical calculus, the fundamental theorem and Taylor's theorem for example. We have seen that there is a very good explanation, within the narrow boundaries of the Nila approach to calculus – they were not needed – for the absence of both, but that emerges only after a reading of all that is in *Yuktibhāṣā*.

It is a legitimate goal of the study of sciences of the past to try and identify when and through what processes a radically new discipline or subdiscipline took form and established itself as such. A first criterion, of course, is that it must, by transcending existing modes of thought, enable the elucidation of hitherto inaccessible questions. But to expect that all future directions of growth should already have been foreseen at birth is both ahistorical and unrealistic. Mādhava has no explicit reference to the idea of a function and the only curves he studied are those of constant curvature, thus obviating the need to consider derivatives higher than the second. It is an exaggeration only in degree to compare that limitation to the impossibility of Newton or Leibniz having envisaged the great generality of the objects which were later to be brought within the realm of calculus – we only have to think of the calculus of distributions (which lie beyond the definition of functions) without which, for example, the theory of partial differential equations will remain incomplete and much of modern physics cannot be formulated. What should be non-negotiable is that the founding principles, in addition to being original, must also be of such scope and robustness as to support the edifice that future generations will build on them. These tests the Nila work passes: what began as elementary calculus with the brilliant idea of linearising at every point a geometrical object that "has curvature" is today a discipline whose versatility neither Mādhava nor Newton nor Leibniz could have visualised. At the end, the historian should rejoice at the rare opportunity that its invention, in two totally different cultures but following such similar paths, offers for insights into mathematical creativity.

Acknowledgements: I give warm thanks to David Mumford for his perceptive comments and queries on a draft of this article and to M. S. Narasimhan for many friendly tutorials on the metaphysics of modern calculus. Most of this work was done while I was the recipient of a Senior Award of the Homi Bhabha Fellowships Council, enjoying the hospitality of the Inter-University Centre for Astronomy and Astrophysics, Pune. To both these institutions I record here my gratitude.

Notes

[1]They are often more broadly designated as belonging to the Kerala school. Almost all of them were residents of a cluster of villages in close mutual proximity around the lower reaches of the river Nila. It may therefore be more appropriate to call the lineage the Nila school as I shall do here, especially since the beginning of astronomy in Kerala dates back to about six centuries

earlier with its centre about a hundred kilometers or more to the south of the Nila.

[2]Malayalam, the language of Kerala, has a few more syllables compared to Sanskrit: the palatal ṟ and ḷ as consonants (not just semi-vowels) and the vowels *e* and *o* in both the short and long versions. The long vowels are marked with an overbar: thus Malayāḷam, Kēraḷa, Niḷā. I have indicated the correct pronunciation of, especially, place names when they occur first, reverting later to the way they are generally written. The same word can, very occasionally, have slightly different 'spellings' in Malayalam and Sanskrit; readers should not be confused by the occurrence of both *parārddham* and *parārdha* for example.

[3]The only other purported general survey in existence [2], published more than seventy years ago, has nothing on geometry in general and on the work done in Kerala during this period in particular. Geometry is the subject of Sarasvati Amma's excellent thesis, later published as the book [3]. It has thorough descriptions of the infinitesimal geometry of the Nila school and is a useful complement to [1].

[4]C. W. Whish in his 1832 presentation of the contents of four texts from Kerala to the Royal Asiatic Society in London [5] had no such doubts and states that the power series came out of the same set of ideas, fluxions and fluents, as they did later for Newton.

[5]For a first assessment of the calculus credentials of *Yuktibhāṣā*, see [6].

[6]A summary account of the evolution of the ideas that culminated in the calculus of the Nila school will be found in [12].

[7]Many of these diagrams can also be found elsewhere, e.g., [3]. Virtually all of them are adaptations of the diagrammatic annotations in the first modern commentary on *Yuktibhāṣā* by Rama Varma (Maru) Tampuran and A. R. Akhilesvara Ayyar [14], published in 1948 and out of print since decades. Like the original, it is in Malayalam but for those who read the language it is an invaluable companion, combining traditional insights with a modern mathematical perspective.

[8]Among the four works that Whish brought to light in his 1832 lecture ([5]) are Putumana Somayāji's *Karaṇpaddhati* (dated possibly 1732) and Śaṅkara Varman's *Sadratnamāla* (dated 1823). Even though they were well separated temporally and geographically (possibly in the case of Putumana, definitely in the case of Śaṅkara Varman) from the heartland of the Nila school, their

affiliation with it cannot be missed; among other things, Śaṅkara Varman extended the computation of π to 18 decimal places by using Mādhava's method of estimating the truncation correction to the π series.

[9]There is a mention in Nīlakaṇṭha's *Āryabhaṭīyabhāṣya* of his having spent his boyhood in a *gurukula* which Kunjunni Raja ([17]) takes to mean the house of his teachers Dāmodara and Ravi, sons of Parameśvara.

[10]Local pronunciation (but not the 'spelling') occasionally drops a syllable, turning it into 'Alatyur' or 'Alattur'. This may account for the place being referred to as 'Ālattūr' by K. V. Sarma in all his writings. There is a much bigger town named Alattur in central Kerala, about a 100 km to the east of Tirunavaya, but as far as anyone knows it has no connection to the Nila school.

[11]Another possibility is that *saṃgama* refers to the confluence of the Nila with the sea. Parameśvara refers more than once to his home being on the Nila and on the shore of the ocean, even using the word *saṃgama* to describe its location. He also says that he is from Aśvatthagrāma, which is generally believed to be the Sanskritisation of Alathiyur. The problem is that Alathiyur lies a few kilometers away from both the river and the sea.

[12]But not into the 17th as suggested by Sarma following some hearsay gathered long ago by Whish ([15,16]), to the effect that Jyeṣṭhadeva was also the author of a narrative work *Dṛkkaraṇa* dated 1607. Apart from the problem of stretching the dates, there are linguistic difficulties with the suggested identification ([6]). It is an amusing thought that there is a solution to the problem of fitting the lifetimes of the chief figures to everything that has survived in the oral tradition: all of them were centenarians.

[13]To confuse the issue further: The critical Malayalam edition of *Yuktibhāṣā* [14] has a foreword by the well known traditional scholar P. Shridhara Menon in which it is said that the true name of Jyeṣṭhadeva is Brahmadatta on the strength of a quotation "*alekhi yuktibhāṣā vipreṇa brahmadattasaṃjñena*" from a copy of the manuscript. It is true that Brahmadatta (like the names of all the other Nila astronomers) is and was a common proper name among Namputiris while Jyeṣṭhadeva is extremely rare if not unique to our author. But Kunjunni Raja ([17]) has argued that Brahmadatta was the name of the scribe who made the copy. The mystery of the name remains.

[14]There is not, and has not been for a long time, a Namputiri house with a name approximating Parangod in the region. In Kerala, and especially in Malabar, a 'house name' functions as a family name; it disappears for

good only when a family becomes extinct which is rare but does happen (as presumably in the case of Nīlakaṇṭha, his natal house Kelallur mana becoming Eta mana).

[15]This point is of some relevance to the poorly understood reasons for the decline of the Nila school. That education was reserved for the brahmins cannot be the whole story because, already by the 13th or 14th centuries, in Kerala it was not. While on the subject it is useful to keep in mind that Nampūtiri is a generic term for brahmins (which they often appended to their personal names) who migrated to Kerala beginning around the 7th-8th centuries CE along the coast from the north. Names like Bhaṭṭatiri, Somayāji, etc. are titles bestowed on Nampūtiris with special skills or achievements not, at the time of interest to us, automatically passed on to the next generation. Emprāntiri as in the case of Mādhava refers to the origin of the family without necessarily implying social inferiority.

[16]There is the instance of a combinatorial formula first presented in Nārāyaṇa's *Gaṇitakaumudi* (mid-14th century) whose asymptotic form gives the coefficients of the sine series and which is mentioned by both Jyeṣṭhadeva and Śaṅkara without reference to either Nārāyaṇa or anyone else. It is also not known where Nārāyaṇa lived.

[17]The last but one verse of *Āryabhaṭīya* (*Golapāda* 49) speaks of "the best of gems that is true knowledge brought up by me from the ocean of true and false knowledge by [means of] the boat of my own intelligence".

[18]K. V. Sarma ([18]) speaks with feeling about his encounter with a ghost manuscript supposedly titled *Golavāda*, "The Theory of the Sphere". (Mādhava was often referred to as *Golavid*).

[19]*Yuktibhāṣā* comes in two parts, with all the mathematics collected together in Part I. Part II describes the astronomical applications. Only Part I is of interest in this article.

[20]See M. D. Srinivas's epilogue ("Proofs in Indian Mathematics") to [1]. These principles were not set down once and for all, but subject to constant reinvestigation. A reading of Nīlakaṇṭha's *Siddhāntadarpaṇa* (and his own commentary on it) and *Jyotirmīmāṃsa* will provide convincing evidence of the importance given to such epistemological issues and their continuing reevaluation. By the end of this article it should become clear, hopefully, that within this framework the logical structure of the proofs is sounder than in Newton's treatment of some of the very same problems.

[21]There are some originalities here, meant mainly to prepare the ground for some of the algebraic (operations with polynomials, rational functions and infinite series) and logical (inductive proofs) innovations that are taken up later. These matters, though not strictly 'calculus', provide indispensable technical support for its formulation and applications.

[22]The concluding sections of chapter 6 are not specifically 'infinitesimal' in nature and will get no further notice here. But they mark the first steps in what might have become a new algebraic direction. These developments are touched upon in [7] and will be described in greater detail in [21].

[23]It seems agreed that this is how Āryabhaṭa obtained his approximate π but starting with the inscribed hexagon rather than the circumscribing square.

[24]Throughout this paper, [YB N.n] will denote section n of chapter N of [4]. Subsection n′ when referred to will be denoted [YB N.n.n′].

[25]It is immediately evident that by integrating up to some $t < 1$ but following exactly the same steps the general arctangent series results. *Yuktibhāṣā* does something very similar but formulated more geometrically (using similar triangles, not surprisingly) to get to the same end. I shall not distinguish between the general and the special cases in what follows. No confusion will arise.

[26]The term used in *Yuktibhāṣā* is *cāpīkaraṇam*, accurately though infelicitously translated as 'arcification'. Statements like "The trigonometric power series of e.g., Gregory and Leibniz seem to have grown out of earlier calculus topics such as . . tangents and normals, . ., and quadratures and rectifications in general which apparently do not figure in these Keralese explorations of the relationships between straight lines and arcs of a circle" [20] are therefore too sweeping. What is true is that the Nila school, unlike Europe with its anchoring in the Greek geometry of conic sections, did not consider curves other than the circle. Their astronomy did not require them to.

[27]Or perhaps Jyeṣṭhadeva had in mind something deeper which he explains later in the chapter (*Yuktibhāṣā* 6.5 and 6.6): the need to ensure that terms in the series must have the numerator ($\sin\theta$) smaller than the denominator ($\cos\theta$). This condition is not met in the second octant.

[28]Associating to an arc half the chord of twice that arc – which from now on I will call simply the half-chord of the arc in line with the Indian custom (*ardhajyā* in Sanskrit, *arddhajyā* as it is written in Malayalam, or *jyārdha*) –

i.e., mapping θ to $\sin\theta$, is the founding step of Āryabhaṭa's trigonometry. As θ tends to 0, the half-chord and the full chord of θ, which is $2\sin(\theta/2)$, tend to one another. For results in the limit, it does not matter which of these two choices is made but for the latter choice *Yuktibhāṣā*'s geometry will have to be redone in a more cumbersome manner.

[29]Indian geometers paid no attention to the general idea of tangency probably because the only conic section they were concerned with was the circle: the tangent to a circular arc at a point is the perpendicular to the radius through that point. But that does not change the fact that determining the differential is the same as finding the slope of the tangent, whether Jyeṣṭhadeva knew it as such or not. Once again, the contrast with Greek geometry is sharp.

[30]The binomial expansion for any but positive integral exponents (where it is finite) seems to be unknown in Indian mathematics despite occasional evocation of the term by modern historians. The general method of recursive refining turns up in many earlier contexts but in the work of the Nila school it acquires a remarkable degree of precision and power ([7]). In particular, the idea of carrying out the refining *ad infinitum* in order to get exact answers represented by infinite series appears to be another of the Nila innovations. For the geometric series occurring here, the result coincides with the binomial expansion.

[31]An obvious point but worth noting is that 'discrete integration by parts' does not translate as the 'discrete Leibniz property': $\delta(a_i b_i)$ is not $a_i\delta b_i + b_i\delta a_i$ but has an extra second order term $\delta a_i \delta b_i$ which vanishes in the limit. *Yuktibhāṣā* has no use for the discrete Leibniz property. It is interesting that Leibniz, whose path to calculus was guided strongly by the discrete fundamental theorem (see for example the writings of H. J. M. Bos, in particular [22]), formulated the property named for him wrongly in his first try.

[32]This is one of the reasons for suggesting that *Yuktibhāṣā*, at least the first part, may have been composed in the early 1520s – by 1530, Nīlakaṇṭha would have been 85 years old (about the same age at which Sarma finished [4]; so the suggestion need not be taken too seriously).

[33]Section 7.2.2 on basic trigonometry already concludes with the observation that, starting with the angle $\pi/4$ and halving it successively, "some" of the sines in the table can be determined. In fact all of them can be found by starting with $\sin\pi/6$ and $\sin\pi/4$, halving the angles successively and utilising the symmetries. As far as the limited aim of making the table is concerned, section 7.4.2 is thus superfluous.

[34]For an appreciation of the originality of the mathematics involved, see Mumford's review of Plofker's book [9].

[35]Particularly striking is the fact that, of the two complementary functions sin and cos, the difference of one is linear in the value of the other at the midpoint. The general result $\sin(\theta + \phi) - \sin(\theta - \phi) = 2 \sin \phi \cos \theta$ follows easily from the addition theorem though proofs of the latter are geometrically more demanding and are attributed to Mādhava. The geometry of the finite difference formula served Nīlakaṇṭha as the starting point of his exact treatment in *Āryabhaṭīyabhāṣya* of the sine table, see [24]. *Yuktibhāṣā* exploits the close connection between the addition formula and the derivative later in the text in an alternative approach to the differential part of the sine series. But it does not use the exact first-difference equations, as Nīlakaṇṭha does, in its treatment of the sine table. This is the second instance, the first being the irrationality of π, of Jyeṣṭhadeva passing over without mention insights found in *Āryabhaṭīyabhāṣya* but not in *Tantrasaṃgraha*.

[36]Compare Leibniz ([22]).

[37]Actually the solvability is not so amazing since we have a set of linear equations for the unknowns $s_4, \cdots, s_{2n}$ in terms of s_1 and s_2. What is remarkable is the method of solution.

[38]The formula is cited in *Yuktibhāṣā* without attribution or proof. This is only the second instance in the book of an important result being used with no proof provided. The formula is restated in Śaṅkara's *Kriyākramakarī* but he too refrains from proving it because "the *yukti* is not easy to follow". The only proof I am aware of uses induction on both i and k ([7]).

[39]The method of recursive substitution employed here is probably the most sophisticated instance of a very general technique that comes in many variants (see [7]), that of successive 'refining' or *saṃskāram*. My highly condensed account of how it is used in the solution of the discrete harmonic equation (though it is not directly 'infinitesimal' in content), taking advantage of modern notational flexibility, is meant to convey the algebraic ingenuity involved and the meticulousness with which details of mathematical reasoning are handled. It is also not fully faithful to the original sequencing of the argument; *Yuktibhāṣā*, as is its style, starts by determining the first two terms in the expansion, then builds up the next higher terms and finally describes the combinatorial formula for the general coefficient. A more detailed annotation respecting *Yuktibhāṣā*'s own order and manner of presentation will be found in [7].

[40]This is not the method by which Newton discovered the series that goes under his name (see section 8). Nor is it the method commonly found in today's textbooks. Indeed, in Europe the idea of turning differential equations into integral equations came in the 19th century, much after Taylor's theorem and the extension of the calculus of the exponential function to complex arguments (the two most popular approaches to the sine series).

[41]This term, used just this once in the whole text, comes nearest to a literal translation of 'infinitesimal'.

[42]The suggestion was made by John Playfair ([25]) a long time ago from the description of the sine table in *Sūryasiddhānta*. *Āryabhaṭīya* itself seems to have been unknown in Europe or America before Kern's edition was published in 1874. Burgess in his edition of *Sūryasiddhānta* (1860) translated the number *aṣṭaśata* in the other name by which *Āryabhaṭīya* was known, *Āryāṣṭaśata* (for the 108 verses in the 3 substantive chapters in the meter *āryā*), wrongly as 800, as did Colebrooke earlier.

[43]Trigonometric ratios were invariably scaled to a standard length in India by multiplying by the radius of the 'unit' circle of circumference $360^o = 21,600'$, namely $R = 21,600'/2\pi = 3438'$ using Āryabhaṭa's value for 2π; thus the terminlogy Rsine for the half-chord (*jyārdha*) etc. used by modern writers. The required division by 2π (and the universal avoidance of a decimal fractional notation in India) may very well be the reason for Āryabhaṭa citing a value for π with reference to a diameter of 20,000 rather than 10,000.

[44]My italics. It is ironic that this appeal to physics, so very rare in Indian astronomy, should be such an absurdity.

[45]The 7th century probably saw a reaction from the orthodox to the radically new thinking of the century of enlightenment, the 6th (Āryabhaṭa himself, Vāgbhaṭa the physician, Bhartṛhari the linguist-philosopher). There is no need here to recall the vicious attacks on Āryabhaṭa by Bhāskara I's contemporary Brahmagupta, generally but not always on doctrinal grounds. The most notorious instance of this relapse into faith-based knowledge was the suppression soon after, by resorting to fraudulent rewriting, of Āryabhaṭa's idea of the spinning earth. Fortunately the revisionists did their work partially and unintelligently, leaving enough trails to Āryabhaṭa's true views for the unbiassed future reader (one of whom may well have been Nīlakaṇṭha) to follow.

[46]Newton writes in a slightly later tract (see [23]) of "the doctrine *recently*

established for decimal numbers". (For the full quote and its context, see [7]). Could it be that unfamiliarity with the "doctrine" came in the way of Europe adopting this most natural way of dealing with vanishingly small (*śūnyaprāya*, of the nature of zero, in *Yuktibhāṣā*'s accurate phrase) quantities? Newton uses decimal arithmetic as a model for the manipulation of infinite series.

[47]To read Newton's earliest notes on calculus (for example "Calculus Becomes an Algorithm" in [29], written in 1663, before the outbreak of the plague) with an eye attuned to *Yuktibhāṣā* is to marvel, first at the facility with which it moves between curves and their equations, but also at the great variety of curves/equations considered.

[48]David Mumford's phrase, personal communication.

[49]Nevertheless, the practical demands of astronomy were often trumped by the delight of pursuing mathematics for its own sake. How else to account for a value of π to 11 decimal places, an accuracy far beyond the needs of astronomy? Likewise, there was no astronomical compulsion for the sine series since the sine table augmented by Mādhava's interpolation formula did the job to the required precision with less labour. The calculus of the area and volume of the sphere makes an even better case as they have no role to play in astronomical calculations at all.

[50]From Brahmagupta until Bhāskara II and later, Indian algebra remained a sort of arithmetic in reverse, a mere means of setting up and solving equations for temporarily undetermined numbers. We can just about begin to glimpse a more abstract algebraic point of view emerging in *Yuktibhāṣā*'s treatment of polynomials and rational functions.

[51]Faith in the power of abstraction seems to have been another casualty of the return to conservatism in the 7th century. Even before Pāṇini (6th-5th century BCE) formally introduced (syntactical) metarules in the study of language, there is very good evidence from the *Veda*s that the sciences of grammar and numbers developed in tandem, the rules governing each influencing the other ([11]). Indeed, it may well be that the first appearance of syntactical "rules without meaning" is (probably) slightly earlier, in the organisation of vedic rituals ([31]). For an eloquent endorsement, much later, by Patañjali (3rd-2nd century BCE) of abstract rule-based methods in the study of language, see [31] (reproduced in [7]). Historically, the last tribute we have to the symbiosis between numbers and words is from the 6th century (CE) linguist-philosopher Bhartṛhari (see [11]).

[52]The article [27] (section 6) has a number of quotations from Bhāskara's *Siddhāntaśiromaṇi* (which incidentally is also the source of the formulae for the sphere) with English translations.

[53]The more precise word for 'instant' is *kṣaṇa* which in fact occurs in one passage but not as a qualifier of "velocity". For the record I add that *tātkālika gati* as an unbroken phrase does not actually occur in any of these passages.

[54]There are other occupational hazards. In the preface to his edition of *Āryabhaṭīya* (1874), Kern writes (about the astronomer Sūryadeva): ". . after the great Bhāskara [II], in an age when the living breath of science had already parted from India" ([32]). This while he was in possession of Whish's collection which included a copy of *Tantrasaṃgraha* and 40 years after Whish's London lecture.

[55]I have discussed this point elsewhere ([7]). The infinitely iterated interpolation formula has about as much to do with calculus as an infinite geometric series has. On the whole, it is difficult to escape the feeling that Śaṅkara, for all his brilliance, was less sensitive to what was truly deep in the work of Mādhava than was Jyeṣṭhadeva.

References

[1] Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton (2008).

[2] Bibhutibhushan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics*, Bharatiya Kala Prakashan, Delhi (Reprint, 2001).

[3] T. A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass Publishers, Delhi (1979).

[4] *Gaṇita-Yukti-Bhāṣā* of Jyeṣṭhadeva, Vol. I, Malayalam text critically edited with English translation by K. V. Sarma, Hindustan Book Agency, New Delhi (2008).

[5] C. W. Whish, "On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the

four *śāstra*s Tantrasamgraha, Yuktibhasha, Caranapaddhati and Sadratnamala", Transactions of the Royal Asiatic Society of Great Britain and Ireland **3**, 509 (1835).

[6] P. P. Divakaran, "The First Textbook of Calculus: *Yuktibhāṣā*", J. Ind. Philos. **35**, 417 (2007).

[7] P. P. Divakaran, "Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics" in C. S. Seshadri (Ed.), *Studies in the History of Indian Mathematics*, Hindustan Book Agency, New Delhi (2010).

[8] Kim Plofker, "Relations between Approximations to the Sine in Kerala Mathematics" in *Contributions to the History of Indian Mathematics*, G. Emch, R. Sridharan and M. D. Srinivas (Eds.), Hindustan Book Agency, New Delhi (2005).

[9] David Mumford, Review of [1], Notices of the Am. Math. Soc. **57**, 385 (2010).

[10] Bhagyashree Bavare, "The Origin of Decimal Counting: Analysis of Number Names in the *Ṛgveda*", Lecture at the International Conference on 'Mathematics in Ancient Times', Kozhikode (August 2010), to be published.

[11] P. P. Divakaran and Bhagyashree Bavare, "Genesis and Evolution of Decimal Counting: Evidence from the *Ṛgveda*", to appear.

[12] P. P. Divakaran, "Calculus in India: The historical and mathematical context", Current Science (Supplement) **99**, 8 (2010).

[13] K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Explanatory notes to chapters 1 to 7 of [4].

[14] *Yuktibhāṣā* Part I, Critical edition with *vyākhyā* (in Malayalam) by Rama Varma (Maru) Tampuran and A. R. Akhilesvara Ayyar, Mangalodayam Ltd., Trissivaperur (1948).

[15] K. V. Sarma, *A History of the Kerala School of Hindu Astronomy*, Vishveshvaranand Institute, Hoshiarpur (1972).

[16] K. V. Sarma, "Āryabhaṭa and Āryabhaṭan astronomy: Antecedents, status and development" in *Proceedings of the International Seminar and Colloquium on 1500 Years of Aryabhateeyam*, Kerala Sastra Sahitya Parishad, Kochi (2002).

[17] K. Kunjunni Raja, *Astronomy and Mathematics in Kerala*, The Adyar Library and Research Centre, Madras (1995) (Originally published in the Adyar Library Bulletin, 1963).

[18] *Tantrasaṅgraha* of Nīlakaṇṭha Somayāji with *Yuktidīpikā* and *Laghu-vivṛti* of Śaṅkara, Critically edited by K. V. Sarma, Vishveshvaranand Vishva Bandhu Institute of Sanskrit and Indological Studies, Hoshiarpur (1977).

[19] K. V. Sarma and S. Hariharan, "*Yuktibhāṣā* of Jyeṣṭhadeva", Ind. J. Hist. Sci. **26**(2), 185 (1991).

[20] Kim Plofker, "Mathematics in India" in Victor Katz (Ed.) *The Mathematics of Egypt, Mesopotamia, China, India and Islam*, Princeton University Press, Princeton (2007).

[21] P. P. Divakaran, "Notes on *Yuktibhāṣā*: Arithmetic and Algebra", in preparation.

[22] H. J. M. Bos, "The Fundamental Concepts of Leibnizian Calculus" in H. J. M. Bos, *Lectures in the History of Mathematics*, American Mathematical Society and London Mathematical Society (1991).

[23] D. T. Whiteside (Ed.), *The Mathematical Papers of Isaac Newton* (Vol. III), The University Press, Cambridge (1968).

[24] *Āryabhaṭīya* of Āryabhaṭa, Critically edited with English translation and commentary by Kripa Shankar Shukla in collaboration with K. V. Sarma, Indian National Science Academy, New Delhi (1976).

[25] John Playfair, "Observations on the Trigonometrical Tables of the Brahmins", Trans. Royal Soc. Edinburgh **4**, 84 (1795).

[26] Agathe Keller, *Expounding the Mathematical Seed*, Birkhäuser Verlag, Basel (2006).

[27] K. Ramasubramanian and M. D. Srinivas, "Development of Calculus in India" in C. S. Seshadri (Ed.), *Studies in the History of Indian Mathematics*, Hindustan Book Agency, New Delhi (2010).

[28] The *Śulbasūtra*s of Baudhāyana, Āpastamba, Kātyāyana and Mānava, Edited with English translation and commentary by S. N. Sen and A. K. Bag, Indian National Science Academy, New Delhi (1983).

[29] D. T. Whiteside (Ed.), *The Mathematical Papers of Isaac Newton* (Vol. I), The University Press, Cambridge (1967).

[30] D. T. Whiteside (Ed.), *The Mathematical Papers of Isaac Newton* (Vol. II), The University Press, Cambridge (1968).

[31] Frits Staal, *Ritual and Mantras: Rules without Meaning* (Indian edition), Motilal Banarsidass Publishers, Delhi (1996).

[32] The *Âryabhaṭîya*, a manual of astronomy with the Commentary *Bhaṭadîpikâ* of Paramâdîśvara, Edited by H. Kern, E. J. Brill, Leiden (1874).